Extra Credit Rocks

Sign up for a Discover® Student Card today and enjoy:

- 0% Intro APR* on Purchases for 6 Months
- No Annual Fee
- Easiest Online Account Management Options
- Full 5% *Cashback Bonus*®* on Get More purchases in popular categories all year
- Up to 1% *Cashback Bonus*®* on all your other purchases
- Unlimited cash rewards that never expire as long as you use your Card

APPLY NOW

DISCOVER
CARD

*View Discover® Card Rates, Fees, Rewards and Other Important Information.

Book Team
Editor *Terry Wesner*
Developmental Editor *Terry Baker*
Production Editor *Mary Klein*

Bernard J. Klein Publishing

President *Joanne Klein*
Chief Executive Officer *Mary Ann Klein-Wesner*
Vice President, Publisher *Timothy J. Wesner*
Vice President, Director of Sales and Marketing *Robert S. Wesner*
Vice President, Director of Production *Thomas S. Wesner*

Cover credit: Terry Wesner

Production, illustrations, photo research, and composition by
Publication Services, Inc.

The credits section for this book is considered an extension of the
copyright page

GetMath Educational Software http://www.getmath.com

ISBN 1-932661-80-8

Printed in the United States of America by Bernard J. Klein Publishing

10 9 8 7 6 5 4

Campfire queen
Cycling champion
Sentimental geologist*

Learn more about Marjon Walrod and tell us more about you. Visit pwc.com/bringit.

Your life. You can bring it with you.

*connectedthinking

PricewaterhouseCoopers

Contents

C H A P T E R 5

Trigonometric Equations

C H A P T E R 6

Oblique Triangles and Vectors

C H A P T E R 7

Complex Numbers and Polar Coordinates

College Loan Corporation

CLC

Need more money for college expenses?

The CLC Private Loan℠ can get you up to **$40,000** a year for college-related expenses.

Here's why the CLC Private Loan℠ is a smart choice:

- ☑ Approved borrowers are sent a check within four business days
- ☑ Get $1000 - $40,000 each year
- ☑ Interest rates as low as prime + 0% (8.66% APR)
- ☑ Quick and easy approval process
- ☑ No payments until after graduating or leaving school

CLICK HERE

or call **800.311.9615.**

We are available 24 hours a day, 7 days a week.

8.66% APR is based on a Prime Rate of 8.25% as of August 1, 2006.

C H A P T E R

2 The Trigonometric Functions

In this chapter we widen the application of trigonometric concepts to include angles of any degree measure. We then learn about another method of angle measurement, called radian measure. This is the system of measurement most often used in higher mathematics, engineering, and the sciences. We begin by reviewing the idea of function.

2–1 Functions

The concept of a function is basic to higher mathematics. It provides a way to describe, or model, many real-world situations. For example, the temperature of a metal bar, heated at one end, varies with the distance from the heated end. We say that the temperature along the bar is a function of the distance along the bar. The number of pounds of tomatoes sold in a given geographic area may vary with the retail price per pound. We say that the number of pounds sold is a function of retail price. In short, any time a change in one measurable quantity can be linked to a change in another measurable quantity, the idea of function can be used to give precise meaning to the idea.

A useful definition of function is the following:

> **Function**
> A function is a set of ordered pairs having the property that no first element of the ordered pairs repeats.

For example, $f = \{(1,3), (4,9), (-2,6)\}$ is a function since, first, it is a set of ordered pairs and second, no first element of these ordered pairs repeats—that is, they are all different. However, $g = \{(1,3), (4,9), (4,6)\}$ is not a function since, although it is a set of ordered pairs, one of the first elements (4) is repeated.

If the weight of some type of steel bar is 1.5 pounds per foot and the bar comes in 1-, 2-, 5-, 8-, and 10-foot lengths, then a function that describes the weight of a bar would be $\{(1,1.5), (2,3), (5,7.5), (8,12), (10,15)\}$, where, of course, the first element of each pair is the length of the bar and the second element is the weight.

The set of all first elements of the ordered pairs in a function is called the **domain** of the function, and the set of all second elements is called the **range** of the function. The domain of f (above) is $\{-2,1,4\}$, and the range of f is $\{3,6,9\}$.

One to one
A function is said to be one to one if no second element of the ordered pairs repeats.

For example, the function f mentioned above is one to one, whereas the function $h = \{(1,5), (2,9), (3,9)\}$ is not one to one since there is a second element, 9, of the ordered pairs that repeats.

Example 2–1 A

For each set of ordered pairs listed,

a. State whether the set is a function or not.
b. If a function, state the domain and range.
c. If a function, state whether it is one to one or not.

1. $\{(-2,8), (5,9), (100,19)\}$
a. The set is a function since no first element repeats.
b. The domain is $\{-2, 5, 100\}$, and the range is $\{8, 9, 19\}$.
c. It is one to one since no second element repeats.

2. $\{(-20,-10), (-3,0), (-3,1), (22,15)\}$
a. The set is not a function since the first element -3 repeats.

3. $\{(-5,3), (-1,9), (12,9), (256,256)\}$
a. The set is a function since no first element repeats.
b. The domain is $\{-5, -1, 12, 256\}$, and the range is $\{3, 9, 256\}$.
c. The function is not one to one because the second element 9 is repeated.

If we reverse the first and second elements in each ordered pair of a one-to-one function f we get a new function. For example, consider the function f.

$$f = \{(2,3), (5,9), (7,16)\}$$

If we reverse the ordered pairs, we get

$$\{(3,2), (9,5), (16,7)\}$$

which is also a function. If we reverse the ordered pairs of the function

$$h = \{(1,5), (2,9), (3,9)\}$$

we get

$$\{(5,1), (9,2), (9,3)\}$$

which is not a function (because one of the first elements, 9, is repeated). This first element repeated because a second element repeated in function h. Note that h was not a one-to-one function.

Now consider the situation in a general way. If we reverse the elements of each ordered pair in a one-to-one function, then no first element in the resulting set will repeat since no second element in the original function was repeated. Thus, reversing the ordered pairs of a one-to-one function produces a function.

If we reverse the elements of each ordered pair in a function that is not one to one, the resulting set cannot be a function, since if the function was not one to one there was a second element that repeated, which becomes a repeated first element in the resulting set. Taken together these statements prove the following theorem:

Theorem
Reversing the elements of the ordered pairs of a function produces a function if and only if the function is one to one.

We call the function produced by reversing the ordered pairs of a one-to-one function f the **inverse function** f^{-1}.

Note A symbol for function with a superscript "-1" does not mean the same thing as an expression with an exponent "-1." Although 3^{-1} means $\frac{1}{3^1}$ or $\frac{1}{3}$, and x^{-1} means $\frac{1}{x}$, the symbol f^{-1} does *not* mean $\frac{1}{f}$ if f represents a function.

■ ***Example 2–1 B***

In each of the given sets of ordered pairs,

a. Determine if the set is a function.
b. If a function, state the domain and range.
c. If a one-to-one function, state its inverse function.

1. $f = \{(1,3), (1,5), (2,5), (4,9)\}$
a. f is not a function since a first element, 1, repeats.

2. $g = \{(-2,-8), (0,2), (2,3), (8,8)\}$
a. g is a function since no first element repeats.
b. The domain of g is $\{-2, 0, 2, 8\}$, and the range is $\{-8, 2, 3, 8\}$.
c. g is one to one since no second element repeats. Therefore, it has an inverse.

$$g^{-1} = \{(-8,-2), (2,0), (3,2), (8,8)\}$$

3. $h = \{(-1,0), (0,0), (1,2)\}$
a. h is a function since no first element repeats.
b. The domain of h is $\{-1, 0, 1\}$, and the range is $\{0, 2\}$.
c. h is not one to one because there is a second element, 0, which repeats. Therefore, h does not have an inverse function. ■

Function notation

We often describe an ordered pair of a function by using "f of x," or "$f(x)$" notation. For example, in the function f is defined as $f = \{(-2,6), (1,3), (4,9)\}$ we would say

$f(-2) = 6$	"f of -2 is 6"
$f(1) = 3$	"f of 1 is 3"
$f(4) = 9$	"f of 4 is 9"

Thus, $f(x)$ notation is a way of describing what range element is associated with a given domain element.

Note $f(-2)$ is read "f of -2" or "f at -2." Also, letters other than "f" can be used. We could write "$g(x)$" or "$h(x)$," for example.

■ ***Example 2–1 C***

In the function $f = \{(-100,10), (-50,20), (0,30)\}$ state

a. $f(-50)$ **b.** $f(10)$

a. $f(-50)$ is 20, since the range element associated with -50 is 20.

b. $f(10)$ does not exist. This is because 10 is not in the domain, so we have no way to relate it to some element in the range. ■

Since most functions contain an infinite number of ordered pairs, we cannot describe them with a list. In these cases we use a rule. The rule is usually combined with $f(x)$ notation. For example, we might describe a function f with the rule

$$f(x) = 5x - 3$$

This rule tells us that to form an ordered pair that belongs to the function, where the domain element is x, compute $5x - 3$. This is the range element. If $x = 2$, then $5x - 3$ becomes $5(2) - 3 = 7$, so the ordered pair $(2,7)$ is in the function f. Usually we write

$$\begin{aligned} f(x) &= 5x - 3 \\ f(2) &= 5(2) - 3 \\ f(2) &= 7 \end{aligned}$$

The statement $f(2) = 7$, verbalized "f of 2 is 7," means that for the domain element 2 the range element is 7.

■ ***Example 2–1 D***

1. If a function f is described by the rule

$$f(x) = -3x + 1$$

form the ordered pairs in f for the domain elements (a) -2, (b) 3, and (c) 5.

a. $f(-2) = -3(-2) + 1$
$f(-2) = 7$, so $(-2,7)$ is an ordered pair in f.

b. $f(3) = -3(3) + 1$
$f(3) = -8$, so $(3,-8)$ is an ordered pair in f.

c. $f(5) = -14$, so $(5,-14)$ is an ordered pair in f.

2. If a function g is described by the rule

$$g(x) = x^2 - 2x + 3$$

form the ordered pairs in g for the domain elements (a) -5, (b) $\sqrt{2}$, and (c) 10.

a. $g(-5) = (-5)^2 - 2(-5) + 3$
$g(-5) = 38$, so $(-5,38)$ is an ordered pair in g.

b. $g(\sqrt{2}) = (\sqrt{2})^2 - 2(\sqrt{2}) + 3$
$g(\sqrt{2}) = 2 - 2\sqrt{2} + 3 = 5 - 2\sqrt{2}$,
so $(\sqrt{2},5 - 2\sqrt{2})$ is an ordered pair in g.

c. $g(10) = 10^2 - 2(10) + 3$
$g(10) = 83$, so $(10,83)$ is an ordered pair in g. ■

In these examples we never stated exactly what the domain of each function was. If we are not told what the domain is, we always use all real numbers for which the rule makes sense. This is called the **implied domain.**

For example, if the rule for a function f were $f(x) = \dfrac{3}{x-2}$, then the domain would be every real number except 2, since $f(2)$ would be $\dfrac{3}{2-2} = \dfrac{3}{0}$, which is undefined. If $f(x) = \dfrac{x}{x^2-1}$ were the rule that described a function, the domain would be all the real numbers except ± 1, since either value would make the denominator of the expression 0.

The trigonometric ratios are functions in the sense of our definition. They can be viewed as sets of ordered pairs in which no first element repeats. For example the sine ratio can be described as the ordered pairs (degree measure of angle, sine of angle). It would include, for example, the ordered pairs $\left(30°,\dfrac{1}{2}\right)$, $\left(45°,\dfrac{\sqrt{2}}{2}\right)$, $\left(60°,\dfrac{\sqrt{3}}{2}\right)$, etc.

Mastery points

Can you

- State the definition of a function?
- Determine if a set is a function?
- State whether a function is one to one?
- State the inverse of a one-to-one function?
- Use $f(x)$ notation to form ordered pairs in a function?

Exercise 2–1

In each of the following sets of ordered pairs:

a. Determine if the set is a function.
b. If a function, state the domain and range.
c. If a one-to-one function, state the inverse.

1. $f = \{(3,5), (4,5), (6,9), (7,10)\}$
2. $g = \{(-4,-3), (-1,1), (1,3), (2,5)\}$
3. $h = \{(-2,-2), (3,4), (4,3)\}$
4. $f = \{(0.5,2), (1.5,3), (2,4), (2.5,5)\}$

5. $g = \{(1,4), (1,5), (5,9), (6,10)\}$

6. $h = \{(-10,-5), (10,5), (12,20), (20,30)\}$

7. $f = \{(1,1), (2,2), (3,3), (4,4)\}$

8. $g = \{(-3,5), (5,8), (8,13), (13,21)\}$

Assume $f = \{(-3,6), (-2,9), (-1,0), (0,19), (4\frac{1}{2},7), (2\pi,11), (200,220), (300,7)\}$ in problems 9 and 10.

9. Find
a. $f(-2)$ **b.** $f(0)$ **c.** $f(2\pi)$ **d.** $f(7)$ **e.** $f(250)$

10. Find
a. $f(-1)$ **b.** $f(4\frac{1}{2})$ **c.** $f(\pi)$ **d.** $f(300)$

In each of the following problems a rule that describes a function is given. Form the ordered pairs that this function contains for the following domain elements:
a. -2 **b.** 0 **c.** $\sqrt{3}$ **d.** $\frac{1}{2}$ **e.** 5

11. $f(x) = 5x - 3$

12. $g(x) = 2 - 3x$

13. $h(x) = (x - 3)(x + 2)$

14. $f(x) = x^2 - 2x + 3$

15. $g(x) = x^2 + x - 1$

16. $h(x) = x^2$

17. $f(x) = 3x^4 - x^2 + 2$

18. $g(x) = 1 - x^2$

19. $h(x) = \dfrac{x}{x + 3}$

20. $f(x) = \dfrac{4}{x^2 - 1}$

21. The statement $t(x) = 500 - 2x$ states the functional relationship between the temperature t of an iron rod at a point x centimeters from the heated end. Find the temperature (in degrees centigrade) for points that are (a) 3 cm, (b) 12 cm, and (c) 104 cm from the heated end.

2–2 *The trigonometric functions—definitions*

There are many situations where we have to think of angles as being nonacute. This is often a situation in which we wish to describe an amount of rotation. For example, a ship may turn through an angle of 215°, a computed tomography (CT) scanner used in medical diagnosis may move through an angle of 360°, or a surveyor may find the measure of the angle at one corner of a piece of land to be 165°20′. For these situations, we often place the angle in a rectangular (x-y) coordinate system.

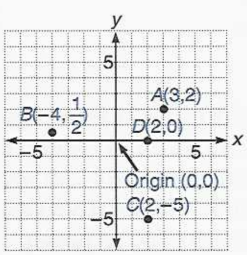


Figure 2–1

The *x-y* (rectangular) coordinate system

We graph using the x-y rectangular coordinate system. Recall that an **ordered pair** is a pair of numbers listed in parentheses, separated by a comma. In the ordered pair (x,y) x is called the **first component** and y is called the **second component;** $(5,-3)$, $(9,3)$, and $(4,\frac{2}{3})$ are examples of ordered pairs. The graphing system we use is formed by sets of vertical and horizontal lines; one vertical line is called the y-axis, and one horizontal line is called the x-axis. The geometric plane (flat surface) that contains this system of lines is called the **coordinate plane.** See figure 2–1.

The **graph** of an ordered pair is the geometric point in the coordinate plane located by moving left or right, as appropriate, according to the first component of the ordered pair, and vertically a number of units corresponding to the second component of the ordered pair. The graphs of the points $A(3,2)$, $B(-4,\frac{1}{2})$, $C(2,-5)$, and $D(2,0)$ are shown in the figure. The first and second elements of the ordered pair associated with a geometric point in the coordinate plane are called its **coordinates.**

Angles in standard position

> **Angle in standard position**
> An angle in standard position is formed by two rays, one of which always lies on the nonnegative portion of the x-axis. This ray is called the **initial side.** The second ray is called the **terminal side.** It may be in any quadrant or along any axis.

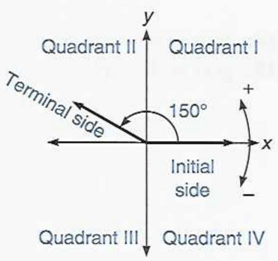


Figure 2–2

Figure 2–2 shows an angle in standard position, with measure 150°. Observe the labeling of the quadrants formed by the x-axis and the y-axis. They are called quadrants I through IV as shown. We generally use the word ''angle'' instead of the phrase ''angle in standard position.''

If the measure of the angle is *positive,* we picture the terminal side as having moved away from the initial side in a *counterclockwise* direction; if the measure of the angle is *negative* we picture the terminal side as having moved away from the initial side in a *clockwise* direction. If an angle's measure is greater than 360° or less than −360° we consider the angle to have ''gone around'' more than once. Several examples of angles in standard position are shown in figure 2–3. In part c we show the angle as a 360° revolution, followed by an additional 200° turn.

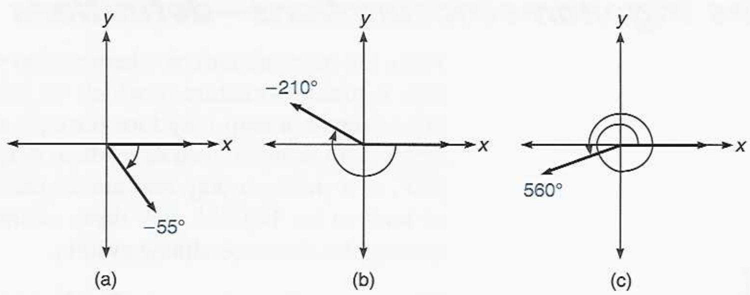


Figure 2–3

Angles which have the same terminal side are said to be **coterminal.** (All angles in standard position have the same initial side.) The 150° angle in figure 2–2 and the −210° angle in figure 2–3 (b) are coterminal. We can see this when we realize that in each case the angle formed by the negative side of the x-axis and the terminal side of each angle is 30°. Since ±360° represents one complete revolution, coterminal angles are angles whose degree measures differ by an integer multiple of 360°. This forms the basis for our definition.

> **Coterminal angles**
> Two angles[1] α and β are said to be coterminal if
> $$\alpha = \beta + k(360°),\ k \text{ an integer.}$$
> ***Concept***
> Two angles are coterminal if the difference of their degree measures is evenly divisible by 360°.

[1]Remember, α is the Greek letter alpha, and β is the Greek letter beta.

■ *Example 2–2 A*

In each case find a coterminal angle with measure x such that $0° \leq x < 360°$.

1. 875°

$875° - 360° = 515°$ Subtract 360° until x is found

$515° - 360° = 155°$ The required angle is 155°

We could have done this more elegantly by computing $875° - 2(360°)$.

2. $-1{,}000°$

$-1{,}000° + 360° = -640°$ Add 360° until x is found

$-640° + 360° = -280°$

$-280° + 360° = 80°$

Or solve by computing $-1{,}000° + 3(360°) = -1{,}000° + 1{,}080° = 80°$. ■

The trigonometric functions

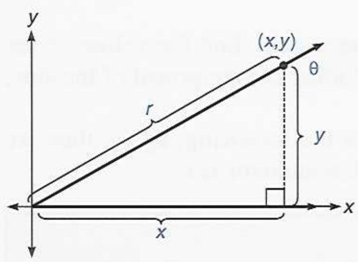


Figure 2–4

We now define the six trigonometric functions. They have the same names as the six trigonometric ratios, and the same abbreviations. The trigonometric ratios are functions with domain the set of *acute* angles. The trigonometric functions have the set of *all* angles as their domain. We used the word ratio to distinguish the two sets of functions. For acute angles the trigonometric functions are essentially the same as the trigonometric ratios. The following definition refers to figure 2–4.

The trigonometric functions

Let θ be an angle in standard position, and let (x,y) be any point on the terminal side of the angle, except (0,0). Let $r = \sqrt{x^2 + y^2}$ be the distance from the origin to the point. Then,

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{r}{y}, \quad \sec\theta = \frac{r}{x}, \quad \cot\theta = \frac{x}{y}$$

Note
1. We define r so that $r > 0$.
2. If x or y in the point (x,y) is zero, then those ratios with x or y in the denominator are not defined.
3. Unlike the trigonometric ratios, the trigonometric functions can take on negative values.
4. We sometimes call the cosecant, secant, and cotangent functions the *reciprocal trigonometric functions.*

It can be proven that for a given angle, *it does not matter what point on the terminal side is chosen; the values of the trigonometric functions will be the same.* This is illustrated in example 2–2 D.

It can also be seen that *coterminal angles have the same values for the trigonometric functions.* This is because two coterminal angles have the same terminal side, and the definitions depend solely on a point on the terminal side.

The definitions of the trigonometric functions imply the following identities for all values of θ for which any denominator is nonzero. These identities look identical to those for the trigonometric ratios; however, those were shown to be true only for acute angles in right triangles.

Reciprocal function identities

$$\csc\theta = \frac{1}{\sin\theta}, \quad \sin\theta = \frac{1}{\csc\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}, \quad \cos\theta = \frac{1}{\sec\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}, \quad \tan\theta = \frac{1}{\cot\theta}$$

To see that the first reciprocal function identity is true observe that $\csc\theta = \frac{r}{y} = \frac{1}{\frac{y}{r}} = \frac{1}{\sin\theta}$. It is left as an exercise to show that the rest of these are true.

Using the reciprocal function identities we can usually find the values of the cosecant, secant, and cotangent functions by finding the reciprocal of the sine, cosine, and tangent functions.

Two other identities that can be useful are the following; again, they are true only for those values of θ for which no denominator is 0.

Tangent/cotangent identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

■ ***Example 2–2 B***

Show that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ is an identity for the trigonometric functions.

We show that each member of the equation is equivalent to the same thing.

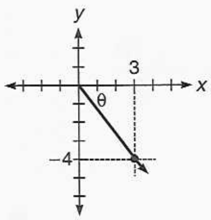
$\tan\theta$	$\frac{\sin\theta}{\cos\theta}$	
$\frac{y}{x}$	$\frac{\frac{y}{r}}{\frac{x}{r}}$	Apply the definitions
	$\frac{y}{r}\cdot\frac{r}{x}$	Algebra of division
	$\frac{y}{x}$	Reduce $\frac{ry}{rx}$

Thus, $\tan\theta = \frac{y}{x}$ and $\frac{\sin\theta}{\cos\theta} = \frac{y}{x}$, so $\tan\theta = \frac{\sin\theta}{\cos\theta}$. ■

Example 2–2 C illustrates finding values of the trigonometric functions for an angle in standard position, given a point on the terminal side of the angle.

■ Example 2–2 C

In each problem a point on the terminal side of an angle θ is given. Use it to find the trigonometric functions for that angle. Also, make a sketch of the angle.



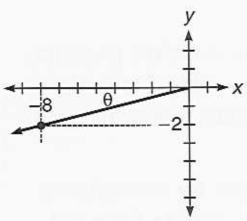
1. (3,−4)

$$r = \sqrt{x^2 + y^2} \qquad \text{Definition}$$
$$= \sqrt{3^2 + (-4)^2} \qquad \text{Replace } x, y$$
$$= 5$$

$$\sin\theta = \frac{y}{r} = -\frac{4}{5},\ \csc\theta = \frac{1}{\sin\theta} = -\frac{5}{4}$$

$$\cos\theta = \frac{x}{r} = \frac{3}{5},\ \sec\theta = \frac{1}{\cos\theta} = \frac{5}{3}$$

$$\tan\theta = \frac{y}{x} = -\frac{4}{3},\ \cot\theta = \frac{1}{\tan\theta} = -\frac{3}{4}$$



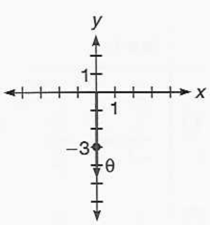
2. (−8,−2)

$$r = \sqrt{x^2 + y^2} \qquad \text{Definition}$$
$$= \sqrt{(-8)^2 + (-2)^2} \qquad \text{Replace } x, y$$
$$= \sqrt{68} = 2\sqrt{17}$$

$$\sin\theta = -\frac{2}{2\sqrt{17}} = -\frac{1}{\sqrt{17}} = -\frac{\sqrt{17}}{17},\ \csc\theta = -\sqrt{17}$$

$$\cos\theta = -\frac{8}{2\sqrt{17}} = -\frac{4}{\sqrt{17}} = -\frac{4\sqrt{17}}{17},\ \sec\theta = -\frac{\sqrt{17}}{4}$$

$$\tan\theta = \frac{-2}{-8} = \frac{1}{4},\ \cot\theta = 4$$



3. (0,−3)

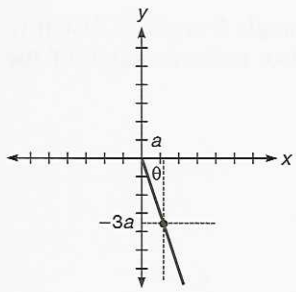
$$r = \sqrt{0^2 + (-3)^2} = 3$$

$$\sin\theta = \frac{y}{r} = \frac{-3}{3} = -1,\ \csc\theta = \frac{1}{\sin\theta} = -1$$

$$\cos\theta = \frac{x}{r} = \frac{0}{3} = 0,\ \sec\theta = \frac{1}{\cos\theta} = \frac{1}{0},\ \text{undefined}$$

$$\tan\theta = \frac{y}{x} = \frac{-3}{0},\ \text{undefined; } \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{0}{-1} = 0$$

As illustrated here, when the tangent function is undefined the cotangent function is defined. In this one case it is useful to use the appropriate tangent/cotangent identity or equivalently $\cot\theta = \dfrac{x}{y}$.



4. $(a,-3a)$, $a > 0$

Since the x-coordinate is positive the angle is in quadrants I or IV; since the y-coordinate is negative, we choose quadrant IV for our sketch.

$$r = \sqrt{a^2 + (-3a)^2} = \sqrt{10a^2} = a\sqrt{10}$$

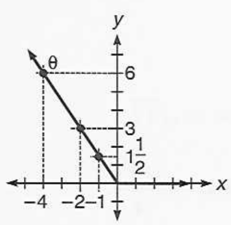
$$\sin\theta = -\frac{3a}{a\sqrt{10}} = -\frac{3}{\sqrt{10}} = -\frac{3\sqrt{10}}{10},\ \csc\theta = -\frac{\sqrt{10}}{3}$$

$$\cos\theta = \frac{a}{a\sqrt{10}} = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10},\ \sec\theta = \sqrt{10}$$

$$\tan\theta = -\frac{3a}{a} = -3,\ \cot\theta = -\frac{1}{3}$$

Example 2–2 D illustrates that the value of the trigonometric functions for a given angle depend only on the measure of the angle and not on the point that is chosen on its terminal side.

Example 2–2 D



The point $(-2,3)$ is on the terminal side of an angle θ in standard position. Find two other points that would also be on the terminal side of this angle and then compute the value of the sine, cosine, and tangent functions with all three points.

We can find other points on the terminal side of this angle by multiplying both values, -2 and 3, by the same amount. Thus, if we double them we obtain the point $(-4,6)$. If we take half of each we obtain $(-1,1\frac{1}{2})$. All three points are shown in the figure.

The computations for the three points are shown in the table. The same results are obtained regardless of which point is used to perform the calculation.

Point	x	y	r	$\sin\theta$	$\cos\theta$	$\tan\theta$
$\left(-1,1\frac{1}{2}\right)$	-1	$\frac{3}{2}$	$\frac{\sqrt{13}}{2}$	$\dfrac{\frac{3}{2}}{\frac{\sqrt{13}}{2}} = \frac{3\sqrt{13}}{13}$	$\dfrac{-1}{\frac{\sqrt{13}}{2}} = -\frac{2\sqrt{13}}{13}$	$\dfrac{\frac{3}{2}}{-1} = -\frac{3}{2}$
$(-2,3)$	-2	3	$\sqrt{13}$	$\frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$	$\frac{-2}{\sqrt{13}} = -\frac{2\sqrt{13}}{13}$	$\frac{3}{-2} = -\frac{3}{2}$
$(-4,6)$	-4	6	$2\sqrt{13}$	$\frac{6}{2\sqrt{13}} = \frac{3\sqrt{13}}{13}$	$\frac{-4}{2\sqrt{13}} = -\frac{2\sqrt{13}}{13}$	$\frac{6}{-4} = -\frac{3}{2}$

Mastery points

Can you

- When given an angle θ, find a positive coterminal angle with measure x such that $0° \le x < 360°$?
- Sketch an angle and find the values of the six trigonometric functions when given a point on the terminal side of the angle?

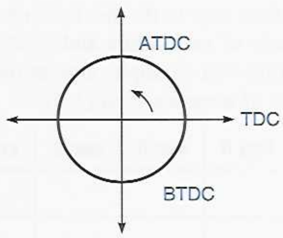
Exercise 2–2

In problems 1–17,

a. Draw the initial and terminal side of the given angle.
b. State the measure of the smallest nonnegative angle that is coterminal with the given angle.

1. 420°	**2.** −40°	**3.** 230°	**4.** 1,000°	**5.** 1,800.6°
6. 1,260°	**7.** 547.9°	**8.** 2,000°	**9.** −870°	**10.** 625°
11. 525°	**12.** −610°	**13.** −1,530.3°	**14.** 390°	**15.** −720°
16. −11.9°	**17.** −313°			

18. An automobile engine is timed to fire the spark plug for cylinder 1 at 8° BTDC (before top dead center), which, for our purposes, is −8°. Assuming this engine rotates in a counterclockwise direction, what is the equivalent amount ATDC (after TDC) (i.e., the least nonnegative angle coterminal with it)?



19. If an automobile engine is timed to fire at 13° BTDC, what is the equivalent amount ATDC?

20. If an automobile engine is timed to fire at 8.6° BTDC, what is the equivalent amount ATDC?

21. If an automobile engine is timed to fire at 6.1° BTDC, what is the equivalent amount ATDC?

22. In an electronic circuit with an inductive component to the impedance, the current follows the voltage. For example, the current may follow the voltage by 15°, in which case we could say the phase angle of the current is −15°, relative to the voltage. We could just as easily say that the phase angle of the voltage is 345°, relative to the current. Find the phase angle of the voltage relative to the current if the phase angle of the current relative to the voltage is (a) −88°, (b) −24.33°, (c) −35°56′, (d) −16.56° (e) −0°14′, (f) −0.14°. (Find the least nonnegative coterminal angle in each case.)

In the following problems you are given a point that lies on the terminal side of an angle in standard position. In each case, compute the value of all six trigonometric functions for the angle.

23. (3,6)	**24.** (−2,5)	**25.** (−5,8)	**26.** (−7,−8)	**27.** (2,−2)
28. (3,0)	**29.** (−1,4)	**30.** (0,−4)	**31.** (−10,−15)	**32.** $(3,\sqrt{5})$
33. $(-\sqrt{2},6)$	**34.** $(3,-\sqrt{6})$	**35.** $(-\sqrt{3},-\sqrt{2})$	**36.** $(1,-\sqrt{3})$	**37.** $(\sqrt{6},-\sqrt{10})$

In the following problems you are given a point that lies on the terminal side of an angle in standard position. In each case, compute the value of all six trigonometric functions for the angle. Assume $a > 0$, $b > 0$.

38. $(b,-2b)$	**39.** $(2a,-a)$	**40.** $(-a,-a)$	**41.** $(\sqrt{2}b,b)$	**42.** $(3a,\sqrt{3}a)$
43. $\left(\frac{b}{2},b\right)$	**44.** $\left(-\frac{a}{3},\frac{a}{2}\right)$			

45. Show that the identity $\sec\theta = \frac{1}{\cos\theta}$ is true, except where $\cos\theta = 0$.

46. Show that the identity $\cot\theta = \frac{1}{\tan\theta}$ is true, except where $\tan\theta = 0$.

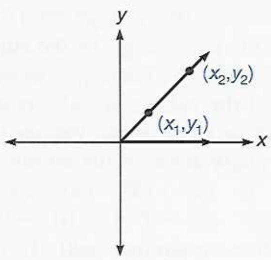
47. Show that the identity $\cot\theta = \frac{\cos\theta}{\sin\theta}$ is true, except where $\sin\theta = 0$.

48. Show that the identity $\cos\theta = \frac{1}{\sec\theta}$ is true.

49. Show that the identity $\sin\theta = \frac{1}{\csc\theta}$ is true.

To solve the following two problems, we must recall that the equation of a nonvertical straight line can be put in the form $y = mx + b$, where m is the slope and b is the y-intercept. If a straight line passes through the origin then $b = 0$ and the equation becomes $y = mx$.

$\boxed{50.}$ Show that if two different points lie on the terminal side of an angle in standard position, then using either point gives the same value for the sine function. For the sake of simplicity assume the terminal side is not vertical or horizontal. Represent the points as (x_1,y_1) and (x_2,y_2). Note that these points lie on the same line. The equation of any line that passes through the origin is of the form $y = mx$, so we know that for the same value of m, $y_1 = mx_1$ and $y_2 = mx_2$. This means that the points (x_1,y_1) and (x_2,y_2) can be rewritten as (x_1,mx_1) and (x_2,mx_2). Use these versions of the points to compute the length r for each point. Then show that the value of the sine function is the same when computed using either point.



51. Show that if the trigonometric function values are the same for two points, then these points lie on the terminal side of the same angle. Assume for simplicity that these points are not located on either the x-axis or the y-axis. To show that the points lie on the terminal side of the same angle we must show that both points lie on the same line and are in the same quadrant. Let (x_1,y_1) and (x_2,y_2) represent the two points, and consider the value of the tangent function as given by each point. This can be used to show that $y_1 = mx_1$ and $y_2 = mx_2$ (for the same value of m). This means that the two points lie on the same line. Now explain why they must be in the same quadrant.

52. Fill in the table below. One way to do this is to choose points on the terminal side of each angle and apply the definitions of each function. For example, a point on the terminal side of an angle of measure 0° is (1,0).

θ	**sin θ**	**cos θ**	**tan θ**	**csc θ**	**sec θ**	**cot θ**
0°						
90°						
180°						
270°						

2–3 Values for any angle—the reference angle/ASTC procedure

The values of the trigonometric functions for an angle of any measure are related to the values for the acute angles of the first quadrant. These values (for the first quadrant) are the same as those for the trigonometric ratios for acute angles. The values of the trigonometric functions for any angle have a sign and a ''size'' (absolute value). We first discuss the sign of the basic trigonometric functions, then the size.

The ASTC rule—the signs of the trigonometric functions by quadrant

The **sign** of the value of a trigonometric function for an angle *depends on the quadrant in which the angle terminates.* Figure 2–5 shows the quadrants in which the sine, cosine, and tangent functions are positive. (They are negative in the other quadrants.)

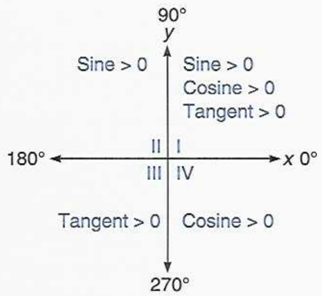


Figure 2–5

The figure shows that the sine function is positive in quadrants I and II, and therefore negative in quadrants III and IV. This is because the sine function is defined by the ratio $\frac{y}{r}$; since r is always positive this ratio is positive where y is positive, in quadrants I and II. Since the cosine function is $\frac{x}{r}$ and $r > 0$, the cosine is positive where x is positive: quadrants I and IV. The tangent function is defined by $\frac{y}{x}$, so it is positive where x and y are both positive (quadrant I) or both negative (quadrant III).

Figure 2–5 should be memorized; it represents the ASTC rule.

The ASTC rule

In quadrant I,	**A**ll the trigonometric functions are positive.
In quadrant II, the	**S**ine function is positive.
In quadrant III, the	**T**angent function is positive.
In quadrant IV, the	**C**osine function is positive.

One memory aid is the sentence "**All Students Take Calculus.**"

Since the sign of the reciprocal of a value is the same as the value, the sign of the cosecant function is the same as the sign of the sine function, that of the secant function is the same as that of the cosine function, and the sign of the cotangent function is the same as that of the tangent function.

The ASTC rule can be used to determine in which quadrant a given angle terminates.

■ ***Example 2–3 A***

Determine in which quadrant the given angle θ terminates.

1. $\sin\theta < 0$, $\tan\theta > 0$

If $\sin\theta < 0$ then θ terminates in quadrants III or IV.
If $\tan\theta > 0$ then θ terminates in quadrants I or III.

Thus, for both conditions to be true, θ must terminate in quadrant III.

2. $\cos\theta < 0$, $\sin\theta > 0$

$\cos\theta < 0$ means θ terminates in quadrant II or III.
$\sin\theta > 0$ means θ terminates in quadrant I or II.

Thus, θ terminates in quadrant II. ■

Reference angles

Angles whose degree measures are integer multiples of 90°, such as 0°, ±90°, ±180°, ±270°, etc. are called **quadrantal angles** because their terminal sides fall between two quadrants. All other angles are nonquadrantal angles. A **reference angle** for a nonquadrantal angle is the acute angle formed by the terminal side of the angle and the x-axis. A reference angle is not defined for

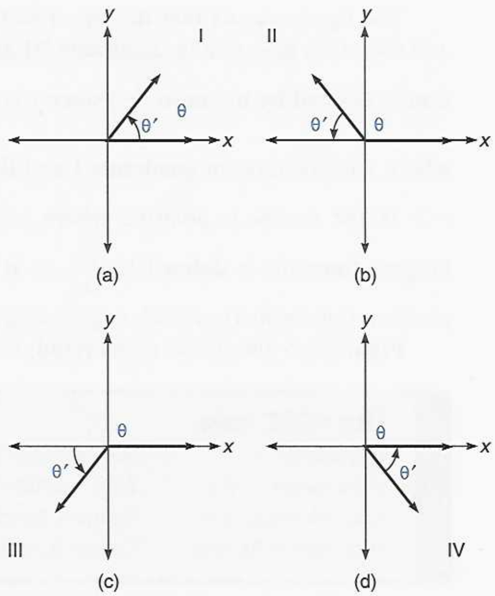


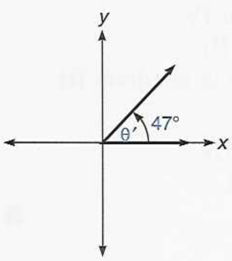
Figure 2–6

quadrantal angles. Figure 2–6 shows a reference angle, θ′ (theta-prime) for an angle θ terminating in each quadrant.

A reference angle is always acute (between 0° and 90°) *and is always formed by the terminal side of the angle and the x-axis* (never the *y*-axis). As will be illustrated in example 2–3 B, a good way to find a reference angle is to sketch the angle itself. This should make clear what computation to perform.

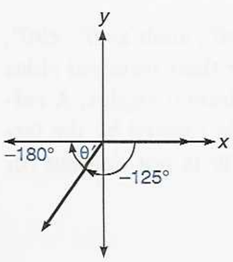
Example 2–3 B

Compute and sketch the reference angle for each angle.



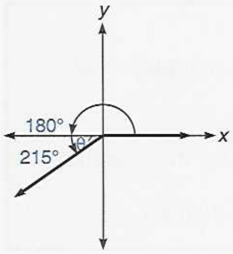
1. 47°

This angle terminates in quadrant I. The reference angle is the same as the angle itself, 47°.

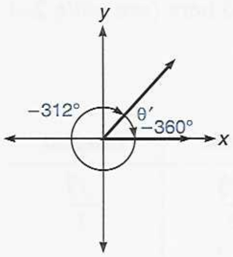


2. −125°

This angle terminates in quadrant III. The positive difference between −125° and −180° is 180° − 125° = 55°, which is the value of the reference angle.



3. 215°
This angle terminates in quadrant III also. Here the reference angle is $215° - 180° = 35°$.



4. −312°
This angle terminates in quadrant I. The value of the reference angle is the positive difference between $-312°$ and $-360° = 360° - 312° = 48°$. ■

It can be seen that if $0° < \theta < 360°$ then the reference angle θ' can be found according to the following formulas.

θ in quadrant I:	$\theta' = \theta$
θ in quadrant II:	$\theta' = 180° - \theta$
θ in quadrant III:	$\theta' = \theta - 180°$
θ in quadrant IV:	$\theta' = 360° - \theta$

The absolute value of the trigonometric functions for any angle

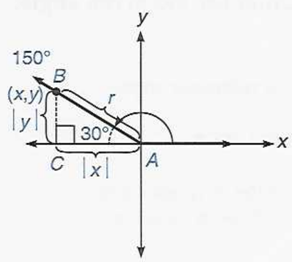


Figure 2–7

The **absolute value** of a trigonometric function for any angle is the same as the trigonometric ratio for the corresponding reference angle. Figure 2–7 illustrates this idea for the angle 150°. If an angle of measure 150° is in standard position, then we find the values of the trigonometric functions by taking a point on its terminal side (the point $B(x,y)$ in the figure), and using the definitions of these functions in terms of x, y, and r.

As seen in the figure, the absolute value of sin 150° is $\frac{|y|}{r}$. This is also the value of the trigonometric ratio for the reference angle, with measure 30°: $\sin 30° = \frac{\text{length of side opposite } 30°}{\text{length of hypotenuse}}$. We know from section 1–3 that $\sin 30° = \frac{1}{2}$. Thus, in absolute value, $|\sin 150°| = \sin 30° = \frac{1}{2}$. Since we know that the sine function is positive in quadrant II, $\sin 150° = \frac{1}{2}$.

The reference angle/ASTC procedure

The facts discussed in the previous paragraphs provide a method for finding the *exact* values of the basic trigonometric functions for any nonquadrantal angle whose reference angle is 30°, 45°, or 60°. We call this the reference angle/ASTC procedure.

> **Reference angle/ASTC procedure**
> To find the exact value of a trigonometric function for a nonquadrantal angle whose reference angle is 30°, 45°, or 60°:
> 1. Find the value of the reference angle.
> 2. Find the value of the appropriate trigonometric ratio for the reference angle from table 1–1, section 1–3.
> 3. Determine the sign of this value using the ASTC rule (figure 2–5).

For convenience, the needed table and figure are repeated here (see table 2–1 and figure 2–8).

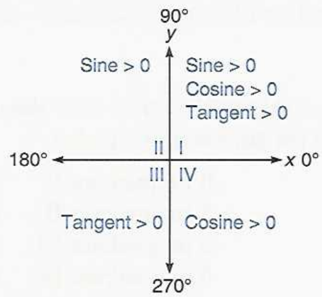


Figure 2–8

	Sine	Cosine	Tangent
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Table 2–1

■ *Example 2–3 C*

Find the exact value of the given trigonometric function for the given angle.

1. $\cos 210°$

$\theta' = 210° - 180° = 30°$ Find the value of the reference angle

$\cos 30° = \dfrac{\sqrt{3}}{2}$ Table 2–1 (memorized value)

$\cos 210° = -\dfrac{\sqrt{3}}{2}$ A 210° angle terminates in quadrant III, where the cosine function is negative

2. $\tan(-45°)$

$\theta' = 45°$ Find the value of the reference angle

$\tan 45° = 1$ Table 2–1 (memorized value)

$\tan(-45°) = -1$ A −45° angle terminates in quadrant IV, where the tangent function is negative

3. $\tan 840°$

$840° - 2(360°) = 120°$ 120° and 840° are coterminal

$\theta' = 180° - 120° = 60°$ Reference angle for 840°

$\tan 60° = \sqrt{3}$ Memorized value

$\tan 840° = -\sqrt{3}$ 840° terminates in quadrant II, where the tangent function is negative ■

The values of the trigonometric functions for quadrantal angles can be found by selecting any point on the terminal side of the angle and using the definitions.

■ ***Example 2–3 D***

Find the values of the six trigonometric functions for the angle with measure 900°.

$900° - 2(360°) = 180°$, so 900° and 180° are coterminal angles.

The point $(-1,0)$ is on the terminal side of a 180° angle, and is therefore on the terminal side of a 900° angle. Use this point to find the values for 900°.

$r = \sqrt{(-1)^2 + 0^2} = 1, x = -1, y = 0.$

$$\sin 900° = \frac{y}{r} = \frac{0}{1} = 0, \csc 900° = \frac{1}{\sin 900°} = \frac{1}{0}\text{; undefined}$$

$$\cos 900° = \frac{x}{r} = \frac{-1}{1} = -1, \sec 900° = \frac{1}{\cos 900°} = \frac{1}{-1} = -1$$

$$\tan 900° = \frac{y}{x} = \frac{0}{-1} = 0, \cot 900° = \frac{1}{\tan 900°} = \frac{1}{0}\text{; undefined}$$

■

Approximate values of the trigonometric functions—calculators

Approximate values of the trigonometric functions are calculated using the same calculator keys as for the trigonometric ratios; for acute angles the ratios and functions have the same values. Recall from section 1–3 that the calculator must be in degree mode when entering angle measure in degrees.

■ ***Example 2–3 E***

Find four decimal place approximations to the following function values.

Make sure the calculator is in degree mode.

1. sin 133° 133 $\boxed{\sin}$ Display $\boxed{0.731353701}$

sin 133° ≈ 0.7314 $\boxed{\text{TI-81}}$ $\boxed{\text{SIN}}$ 133 $\boxed{\text{ENTER}}$

2. tan (−18°) 18 $\boxed{\pm}$ $\boxed{\tan}$ Display $\boxed{-0.324919696}$

tan (−18°) ≈ −0.3249 $\boxed{\text{TI-81}}$ $\boxed{\text{TAN}}$ $\boxed{(-)}$ 18 $\boxed{\text{ENTER}}$

3. sec (−335.6°) $\sec(-335.6°) = \dfrac{1}{\cos(-335.6°)}$

335.6 $\boxed{\pm}$ $\boxed{\cos}$ $\boxed{1/x}$

$\boxed{\text{TI-81}}$ $\boxed{(}$ $\boxed{\text{COS}}$ $\boxed{(-)}$ 335.6 $\boxed{)}$

$\boxed{x^{-1}}$ $\boxed{\text{ENTER}}$

sec (−335.6°) ≈ 1.0981 Display $\boxed{1.098076141}$

■

Solutions to trigonometric equations

Recall from chapter 1 that we use the inverse trigonometric functions to solve trigonometric equations of the form $\sin\theta = k$, $\cos\theta = k$, $\tan\theta = k$, where k is a known constant. In particular, to find one value of θ in each equation, we use the following facts:

if $\sin\theta = k$, then one solution for θ is $\theta = \sin^{-1} k$
if $\cos\theta = k$, then one solution for θ is $\theta = \cos^{-1} k$
if $\tan\theta = k$, then one solution for θ is $\theta = \tan^{-1} k$

In chapters 4 and 5 we will examine this situation in more depth, but for now we will simply rely on these facts, and on the fact that these inverse trigonometric functions are programmed into calculators as seen in section 1–3.

Example 2–3 F

Find one solution to each trigonometric equation, to the nearest 0.1°.

1. $\sin\theta = -0.8500$
   $\theta = \sin^{-1}(-0.8500) \approx -58.2°$
2. $\cos\theta = -0.8500$
   $\theta = \cos^{-1}(-0.8500) \approx 148.2°$

Mastery points

Can you

- Determine in which quadrant an angle terminates when given the signs of two of the trigonometric function values for that angle?
- Compute and sketch the reference angle for a given nonquadrantal angle θ with given degree measure?
- Find the exact value of any trigonometric function for an angle whose reference angle is 30°, 45°, or 60°, using the reference angle/ASTC procedure?
- Find the exact value of any trigonometric function for a quadrantal angle?
- Find the approximate value of any trigonometric function using a calculator?
- Find the approximate value of one solution to an equation of the form $\sin\theta = k$, $\cos\theta = k$, $\tan\theta = k$?

Exercise 2–3

In the following problems you are given the sign of two of the trigonometric functions of an angle in standard position. State in which quadrant the angle terminates.

1. $\sin\theta > 0, \cos\theta < 0$
2. $\sec\theta < 0, \tan\theta > 0$
3. $\cos\theta > 0, \tan\theta > 0$
4. $\cot\theta < 0, \csc\theta > 0$
5. $\tan\theta < 0, \csc\theta < 0$
6. $\sec\theta > 0, \csc\theta < 0$
7. $\csc\theta > 0, \cos\theta < 0$
8. $\tan\theta > 0, \sin\theta < 0$
9. $\sec\theta > 0, \sin\theta < 0$
10. $\cot\theta > 0, \sin\theta > 0$
11. $\sin\theta < 0, \sec\theta < 0$

For each of the following angles, find the measure of the reference angle θ'.

12. $39.3°$
13. $164.2°$
14. $213.2°$
15. $427.1°$
16. $-16.8°$
17. $-255.3°$
18. $-100.4°$
19. $130.7°$
20. $-671.3°$
21. $-181.0°$
22. $512.8°$
23. $-279.5°$
24. $292.3°$
25. $-252°$
26. $312°$

Find the exact trigonometric function value for each angle.

27. $\sin 135°$
28. $\cos 120°$
29. $\sin 210°$
30. $\cos 330°$
31. $\tan 300°$
32. $\sin 240°$
33. $\sin(-120°)$
34. $\cos(-315°)$
35. $\cos 660°$
36. $\csc(-315°)$
37. $\cot 300°$
38. $\sin 450°$
39. $\cos(-450°)$
40. $\tan(-540°)$
41. $\csc 90°$
42. $\sin 840°$
43. $\sin(-690°)$
44. $\cot 215°$
45. $\sec 150°$
46. $\tan 330°$

Find the trigonometric function value for each angle to four decimal places.

47. $\sin 113.4°$
48. $\cos 88.2°$
49. $\tan 214.6°$
50. $\csc 345°10'$
51. $\cot 412°$
52. $\tan 527.2°$
53. $\sec(-13°)$
54. $\sin(-88°)$
55. $\cos(-355°20')$
56. $\tan(-248.6°)$
57. $\csc 285.3°$
58. $\sec 211°$
59. $\cos(-133.2°)$
60. $\sin(-293°50')$

Find one approximate solution to each equation, to the nearest 0.1°.

61. $\sin\theta = 0.25$
62. $\sin\theta = \frac{1}{3}$
63. $\cos\theta = -0.5$
64. $\cos\theta = 0.813$
65. $\tan\theta = -\frac{8}{5}$
66. $\tan\theta = 3$
67. $\sin\theta = -0.59$
68. $\cos\theta = -0.18$

69. In a certain electrical circuit the instantaneous voltage E (in volts) is found by the formula $E = 156\sin(\theta + 45°)$. Compute E to the nearest 0.01 volt for the following values of θ:
a. 0° **b.** 45° **c.** 100° **d.** $-200°$ **e.** 13.3° **f.** $-45°$

70. In a certain electrical circuit the instantaneous current I (in amperes) is found by the formula $I = 1.6\cos(800t)°$. Find I to the nearest 0.01 ampere for the following values of t:
a. 0 **b.** 0.25 **c.** 0.85 **d.** 1 **e.** -1 **f.** -2.5 **g.** -0.02

71. If a force of 200 pounds is applied to a rope to drag an object, the actual force tending to move the object horizontally is $f(\theta) = 200\cos\theta$, where θ is the angle the rope makes with the horizontal. Compute the force tending to move the object horizontally if the angle of the rope is
a. 0° **b.** 25° **c.** 50°

72. If a rocket is moving through the air at a speed of 1,200 mph, at an angle of $\theta°$ with the horizontal, then the rate at which it is rising is $v(\theta) = 1200\sin\theta$. Find the rate at which a rocket moving at 1200 mph is rising if the angle it makes with the horizontal is
a. 50° **b.** 60° **c.** 70° **d.** 80°

73. Use the values 30° and 60° to see if the statement $\sin(2\theta) = 2\sin\theta$ is true. (Let θ be 30°.)

74. Use the values 30° and 60° to see if the statement $\sin\frac{\theta}{2} = \frac{\sin\theta}{2}$ is true. (Let θ be 60°.)

75. Use the values 30°, 60°, 90° to see if the statement $\sin(\alpha + \beta) = \sin\alpha + \sin\beta$ is true.

2–4 Finding values from other values—reference triangles

Finding a general angle from a value and quadrant

In sections 1–3 and 1–4 we learned how to find the degree measure of an acute angle if we know the value of one of the trigonometric ratios for that angle. We used the inverse sine, cosine, or tangent function as appropriate. We are now dealing with angles of any measure, but the same procedure can be used to find the value of a reference angle. From this we can find the least nonnegative measure for an angle.

As is illustrated in example 2–4 A, we *always find a reference angle* θ' *by finding the inverse sine, cosine, or tangent function value for a positive value.* We use a positive value to obtain an acute angle (all reference angles are acute). We could summarize the procedure as follows.

Finding the least nonnegative measure of an angle from a trigonometric function value and information about a quadrant.

1. If necessary use the ASTC rule[2] to determine the quadrant for the terminal side of the angle.
2. Use $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$ to find θ'. Use the absolute value of the given trigonometric function value.
3. Apply θ' to the correct quadrant to determine the value of θ.

Note We find the "least nonnegative value." There are actually an unlimited number of values, since the trigonometric values are the same for all coterminal angles.

In section 2–3 we saw formulas that find θ' if $0° < \theta < 360°$. These formulas can be solved for θ if necessary and thus provide a formula for finding θ given θ'.

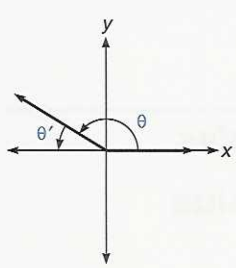
Relationship between θ and θ' if $0° < \theta < 360°$

θ in Quadrant I:	$\theta' = \theta$	$\theta = \theta'$
θ in Quadrant II:	$\theta' = 180° - \theta$	$\theta = 180° - \theta'$
θ in Quadrant III:	$\theta' = \theta - 180°$	$\theta = \theta' + 180°$
θ in Quadrant IV:	$\theta' = 360° - \theta$	$\theta = 360° - \theta'$

It is interesting to observe that the formulas are the "same" when solved for θ and θ' for every quadrant except quadrant III.

Example 2–4 A

Find the least nonnegative measure of θ to the nearest 0.1°.



1. $\sin\theta = 0.5150$ and $\cos\theta < 0$

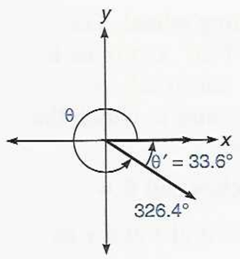
Since $\sin\theta > 0$ and $\cos\theta < 0$, θ terminates in quadrant II (see the figure). We find the acute reference angle θ' just as we did in section 1–2.

$$\theta' = \sin^{-1} 0.5150 \approx 31.0°$$

Thus, $\theta \approx 180° - 31.0° = 149.0°$.

Note The calculator can be used to verify our result by checking that $\sin 149° \approx 0.5150$ and that $\cos 149° < 0$.

[2]Section 2–3.



2. $\tan\theta = -0.6644$ and $\sin\theta < 0$
Since $\tan\theta < 0$ and $\sin\theta < 0$, θ terminates in quadrant IV.

$$\theta' = \tan^{-1} 0.6644 \approx 33.6^\circ$$ Note we use the positive value 0.6644

$$\theta = 360^\circ - \theta' \approx 326.4^\circ$$

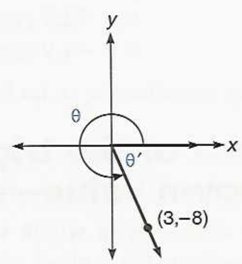
Example 2–4 B applies several of the things we have been studying.

■ ***Example 2–4 B***

The point $(3,-8)$ is on the terminal side of θ.

a. Draw a representation of θ.
b. Find the exact value of the trigonometric functions for θ.
c. Find the least nonnegative measure of θ, to the nearest 0.1°.

a. The representation is shown in the figure.



b. Applying the definitions of section 2–2:

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + (-8)^2} = \sqrt{73}$$

$$\sin\theta = \frac{y}{r} = \frac{-8}{\sqrt{73}} = -\frac{8\sqrt{73}}{73}, \qquad \csc\theta = \frac{1}{\sin\theta} = -\frac{\sqrt{73}}{8}$$

$$\cos\theta = \frac{x}{r} = \frac{3}{\sqrt{73}} = \frac{3\sqrt{73}}{73}, \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{\sqrt{73}}{3}$$

$$\tan\theta = \frac{y}{x} = -\frac{8}{3}, \qquad \cot\theta = \frac{1}{\tan\theta} = -\frac{3}{8}$$

c. We can find θ' by using the fact that $\tan\theta' = |\tan\theta| = \frac{8}{3}$, so that

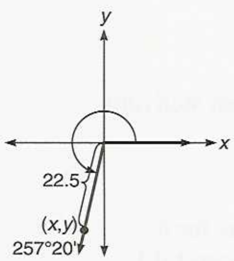
$$\theta' = \tan^{-1}\tfrac{8}{3} \approx 69.4^\circ, \text{ so}$$

$$\theta \approx 360^\circ - 69.4^\circ = 290.6^\circ$$

There are many places in science and technology where we find applications for trigonometric functions. With the advent of numerically controlled, or computer-controlled, machines these applications are becoming more common.

■ *Example 2–4 C*

A technician is setting up a numerically controlled grinding wheel. The starting position for the wheel must be at an angle of 257°20′ and must be 22.5 inches from the origin (assuming the machine uses our usual x-y coordinate system). Find the x- and y-coordinates of the point at which the grinding wheel must start, to the nearest tenth of an inch.



The figure illustrates the situation. We have $r = 22.5$ inches and $\theta = 257°20'$. By definition, $\sin\theta = \dfrac{y}{r}$ and $\cos\theta = \dfrac{x}{r}$, so we find x and y as follows:

$$\sin 257°20' = \frac{y}{22.5}$$
$$y = 22.5 \sin 257°20'$$
$$y \approx -22.0 \text{ inches} \qquad \boxed{\text{CS 1}}$$
$$\cos 257°20' = \frac{x}{22.5}$$
$$x = 22.5 \cos 257°20'$$
$$x \approx -4.9 \text{ inches}$$

Thus, the starting coordinates, in inches, for the grinder are $(-4.9, -22.0)$. ■

Exact values of the trigonometric functions from a known value—reference triangles

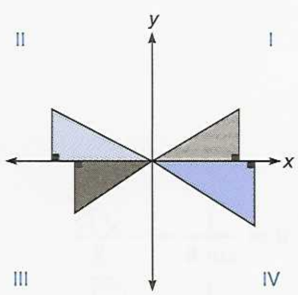
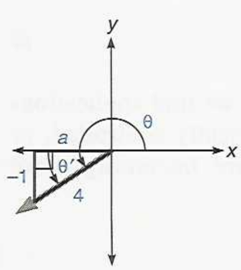


Figure 2–9

There are many situations in which we know the exact value of one of the trigonometric functions for a given angle and need to find the exact value of one or more of the remaining five trigonometric functions for the same angle. We can do this by using a reference triangle, which is a convenient way of combining the idea of reference angle and right triangle. A **reference triangle** is a right triangle with one leg on the x-axis and one leg parallel to the y-axis. The acute angle on the x-axis is the reference angle for the angle in question. *The lengths of the legs of a reference triangle are treated as directed distances (i.e., positive or negative); the hypotenuse is always positive.* This is illustrated in example 2–4 D. Figure 2–9 shows a reference triangle for each quadrant.

■ *Example 2–4 D*

In each case draw a representation of angle θ and use a reference triangle to help find the values of the other five trigonometric functions. Also, find the least positive value of θ to the nearest 0.1°.



1. $\sin\theta = -\frac{1}{4}$ and $\tan\theta > 0$

We know θ terminates in quadrant III since $\sin\theta < 0$ and $\tan\theta > 0$. We construct a right triangle in quadrant III in which one acute angle is a reference angle. This is shown in the figure. We label the hypotenuse 4 and the directed side opposite θ′ as −1. Thus,

$$\sin\theta' = \frac{\text{length of side opposite } \theta'}{\text{length of hypotenuse}} = -\tfrac{1}{4}.$$

$a^2 + (-1)^2 = 4^2$ Find the value of $|a|$ using the Pythagorean theorem; since we are squaring values this theorem works for directed distances

$a^2 = 15$

$a = \pm\sqrt{15}$

We choose $a = -\sqrt{15}$ since it is negative as a directed distance.

We can now use the definitions of the trigonometric ratios for θ' along with the directed distances to find the remaining trigonometric function values for θ.

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = -\frac{\sqrt{15}}{4},\ \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{-1}{-\sqrt{15}} = \frac{\sqrt{15}}{15}$$

$$\csc\theta = \frac{1}{\sin\theta} = -4, \qquad \sec\theta = \frac{1}{\cos\theta} = -\frac{4}{\sqrt{15}} = -\frac{4\sqrt{15}}{15}$$

$$\cot\theta = \frac{1}{\tan\theta} = \sqrt{15}$$

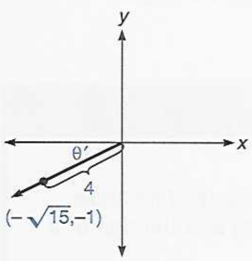
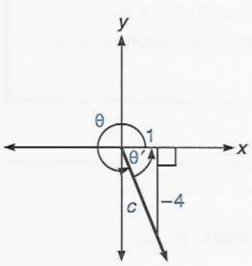


Figure 2–10

We now find an approximation to θ.

$$\sin\theta' = \tfrac{1}{4},\text{ so } \theta' = \sin^{-1}\tfrac{1}{4} \approx 14.5°,\text{ so } \theta \approx 180° + 14.5° = 194.5°.$$

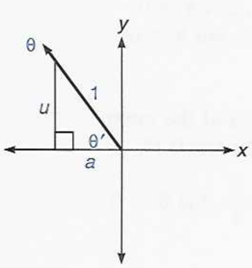
Note The reference triangle works because it is equivalent to finding a point on the terminal side of θ and applying the definitions of the trigonometric functions (section 2–2). The reference triangle above was equivalent to finding the point $(-\sqrt{15}, -1)$ to be on the terminal side of angle θ. (Figure 2–10)



2. $\cot\theta = -\dfrac{1}{4}$ and $270° < \theta < 360°$

If $\cot\theta = -\frac{1}{4}$ then $\tan\theta = -4$. The figure shows a reference triangle for an angle in quadrant IV with tangent -4.

$$c^2 = 1^2 + (-4)^2$$
$$c = \sqrt{17}$$
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{-4}{\sqrt{17}} = -\frac{4\sqrt{17}}{17},\ \cos\theta = \frac{1}{\sqrt{17}} = \frac{\sqrt{17}}{17},$$
$$\csc\theta = \frac{1}{\sin\theta} = -\frac{\sqrt{17}}{4},\ \sec\theta = \frac{1}{\cos\theta} = \sqrt{17},$$
$$\tan\theta' = 4,\text{ so } \theta' = \tan^{-1}4 \approx 76.0°,\text{ and } \theta = 360° - \theta' \approx 284°.$$



3. $\sin\theta = u$ and θ terminates in quadrant II

The figure shows a reference triangle in quadrant II, where $\sin\theta' = u$.

$$u^2 + a^2 = 1^2 \qquad \text{Find the value of side } a$$
$$a = \pm\sqrt{1-u^2}$$
$$a = -\sqrt{1-u^2} \qquad \text{Choose } a < 0 \text{ as a directed distance}$$
$$\cos\theta = \frac{a}{1} = a = -\sqrt{1-u^2},$$
$$\tan\theta = \frac{u}{a} = \frac{u}{-\sqrt{1-u^2}} = -\frac{u}{\sqrt{1-u^2}},$$
$$\sec\theta = \frac{1}{\cos\theta} = -\frac{1}{\sqrt{1-u^2}},\ \csc\theta = \frac{1}{\sin\theta} = \frac{1}{u},$$
$$\cot\theta = \frac{1}{\tan\theta} = -\frac{\sqrt{1-u^2}}{u}$$

Since we do not know the actual value of u we cannot make a determination of an approximate value for angle θ. ■

Calculator steps

1. Ⓐ 22.5 $\boxed{\times}$ $\boxed{(}$ 257 $\boxed{+}$ 20 $\boxed{\div}$ 60 $\boxed{)}$ $\boxed{\sin}$ $\boxed{=}$

   Display $\boxed{-21.9524013}$

   Ⓟ 22.5 $\boxed{\text{ENTER}}$ 257 $\boxed{\text{ENTER}}$ 20 $\boxed{\text{ENTER}}$ 60

   $\boxed{\div}$ $\boxed{+}$ $\boxed{\sin}$ $\boxed{\times}$

   $\boxed{\text{TI-81}}$ 22.5 $\boxed{\times}$ $\boxed{\text{SIN}}$ $\boxed{(}$ 257 $\boxed{+}$ 20 $\boxed{\div}$ 60 $\boxed{)}$

   $\boxed{\text{ENTER}}$

Mastery points

Can you

- Find an approximation to the least nonnegative measure of an angle, given the value of one of the trigonometric functions and the sign of a second for that angle?
- Apply the definitions of the trigonometric functions in appropriate situations?
- Use reference triangles to find the exact values of the remaining trigonometric functions for a given angle, when given the value of one of the trigonometric functions of that angle?

Exercise 2–4

Find the measure of the least nonnegative angle that meets the conditions given in the following problems, to the nearest 0.1°.

1. $\sin\theta = 0.8251$, $\cos\theta > 0$
2. $\cos\theta = -0.1771$, $\sin\theta < 0$
3. $\tan\theta = 0.6569$, $\sec\theta > 0$
4. $\sin\theta = -0.6508$, $\tan\theta > 0$
5. $\sec\theta = -1.0642$, $\sin\theta < 0$
6. $\csc\theta = -1.3673$, $\tan\theta > 0$
7. $\tan\theta = -0.0349$, $\csc\theta < 0$
8. $\cos\theta = -0.2222$, $\sin\theta > 0$
9. $\sin\theta = \frac{3}{8}$, $\cos\theta > 0$
10. $\sin\theta = \frac{3}{8}$, $\cos\theta < 0$
11. $\cot\theta = -5$, $\sin\theta > 0$
12. $\tan\theta = -5$, $\sin\theta < 0$
13. $\cos\theta = -\frac{5}{7}$, $\tan\theta > 0$
14. $\cos\theta = -\frac{5}{7}$, $\tan\theta < 0$

In each case (a) draw a representation of angle θ and (b) use a reference triangle to help find the values of the other trigonometric functions. Also, (c) find the reference angle θ′ and the least positive value of θ to the nearest 0.1°.

15. $\sin\theta = \dfrac{3}{4}$, $\cos\theta > 0$
16. $\sin\theta = \dfrac{4}{5}$, $\cos\theta < 0$
17. $\cos\theta = -\dfrac{1}{2}$, $\tan\theta > 0$
18. $\cos\theta = -\dfrac{5}{13}$, $\tan\theta < 0$
19. $\sin\theta = 1$
20. $\cos\theta = 1$
21. $\tan\theta = 2$, $\cos\theta < 0$
22. $\tan\theta = 3$, $\cos\theta > 0$
23. $\csc\theta = -5$, $\sec\theta < 0$
24. $\csc\theta = -2$, $\sec\theta > 0$
25. $\csc\theta = -1$
26. $\sec\theta = -1$
27. $\sin\theta = -\frac{3}{4}$, $\tan\theta > 0$
28. $\sin\theta = -\frac{2}{5}$, $\tan\theta < 0$
29. $\sec\theta = 4$, $\csc\theta > 0$
30. $\sec\theta = \sqrt{6}$, $\csc\theta < 0$
31. $\cot\theta = \dfrac{\sqrt{2}}{3}$, $\sin\theta < 0$
32. $\cot\theta = \frac{1}{3}$, $\sin\theta > 0$
33. $\cos\theta = -\frac{5}{13}$, $\sin\theta > 0$
34. $\cos\theta = -\dfrac{3}{\sqrt{10}}$, $\sin\theta < 0$
35. $\tan\theta = \frac{7}{2}$, $\sec\theta < 0$

36. $\tan\theta = \frac{7}{3}$, $\sec\theta > 0$

37. $\sec\theta = 5$, $\tan\theta > 0$

38. $\sec\theta = 4$, $\tan\theta < 0$

39. $\sin\theta = \dfrac{1}{\sqrt{5}}$, $\tan\theta < 0$

40. $\sin\theta = \dfrac{1}{\sqrt{3}}$, $\tan\theta > 0$

Solve the following problems.

41. The point $(2,-5)$ is on the terminal side of θ.
  a. Find the exact value of each of the trigonometric functions for θ.
  b. Find the least nonnegative measure of θ, to the nearest 0.1°.

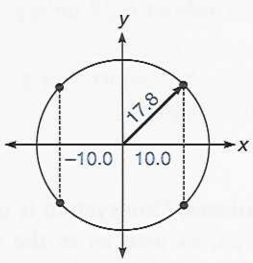
42. The point $(-3,-9)$ is on the terminal side of θ.
  a. Find the exact value of each of the trigonometric functions for θ.
  b. Find the least nonnegative measure of θ, to the nearest 0.1°.

43. A numerically controlled drill is being set up to drill a hole in a piece of steel 6.8 millimeters from the origin at an angle of 135°30′. To the nearest 0.01 millimeter, what are the coordinates of this point?

44. Suppose the hole in problem 43 must be 10.25 inches from the origin at an angle of 13°20′. Find the coordinates of this point to the nearest 0.01 inch.

45. Suppose the hole in problem 43 must be 8.25 centimeters from the origin at an angle of −134.4°. Find the coordinates of this point to the nearest 0.01 centimeter.

46. A numerically controlled drill must drill four holes on a circle whose center is at the origin with radius 17.8 centimeters, as shown in the diagram. The holes must be drilled wherever on this circle the x-coordinate is ±10.0 centimeters. Find the y-coordinate and angle (to the nearest 0.1°) for each of these four holes.



47. Suppose in problem 46 four additional holes must be drilled wherever the y-coordinate is ±15.5 cm. Find the x-coordinate and angle for each of these holes.

48. A technician is aligning a laser device that is used to cut patterns out of cloth. The device is positioned at an angle of 135.20° and at a distance 5.50 feet from the origin. What should the x- and y-coordinates be at this point, to the nearest 0.01 foot?

49. A scanning device used in medical diagnosis has a moving part that moves with great precision in a circle around the patient. Assume the y-axis is perpendicular to the top of the table on which the patient lies and the x-axis is at right angles to the length of the table. The diameter of the machine is 4 feet 3.5 inches. Find the coordinates of the moving part when the angle is 211.5°, to the nearest 0.1 inch.

In problems 50–55 find values of the other five trigonometric functions in terms of u.

50. $\cos\theta = u$ and θ terminates in quadrant I.
51. $\tan\theta = u$ and θ terminates in quadrant I.
52. $\cos\theta = u$ and θ terminates in quadrant III.
53. $\tan\theta = u$ and θ terminates in quadrant III.
54. $\sin\theta = u + 1$ and θ terminates in quadrant I.
55. $\cos\theta = 1 - u$ and θ terminates in quadrant I.

56. In the June 1980 issue of *Popular Science* magazine Mr. R. J. Ransil presented several formulas for calculating saw angles for compound miters. The formulas are:

$$\text{angle } A = 90° - \frac{180°}{\text{number of sides}}$$
$$\tan(\text{arm angle}) = \cot(A)\cdot\sin(\text{slope})$$
$$\sin(\text{tilt angle}) = \cos(A)\cdot\cos(\text{slope})$$

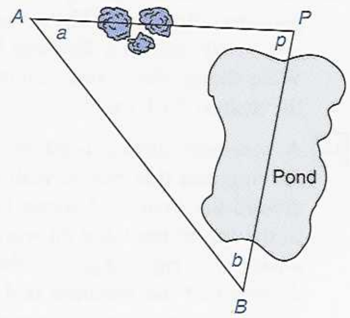
Arm angles and tilt angles are acute.

Calculate the arm angle and tilt angle to the nearest 0.1° for the following numbers of sides and slopes:

Number of sides	Slope (in degrees)
3	5
5	5
7	25
7	30
6	35
8	35

$\boxed{57.}$ A surveying manual describes how to find distance BP in the figure. The distance AP can be found, but trees prevent measuring angle a. Angle b can be measured, but not distance BP. The manual instructs the surveyor to find BP by solving the following sequence of formulas:

$$\sin p = \frac{AB \sin b}{AP}$$
$$a = 180° - (b + p)$$
$$BP = \frac{AP \sin a}{\sin b}$$



Note that the first formula does not give angle p but only $\sin p$. Also, assume p is acute. Solve the sequence of formulas to compute the distance BP to the nearest 0.1 foot if $AB = 512.4$ feet, $AP = 322.6$ feet, and $b = 28.3°$.

58. Find BP in problem 57 to the nearest 0.1 meter if $AB = 319.2$ meters, $AP = 225.7$ meters, and $b = 31.6°$.

2–5 Radian measure—definitions

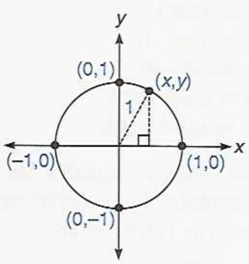


Figure 2–11

The unit circle

The circle with radius one and center at the origin is described by the equation

$$x^2 + y^2 = 1$$

It is called the unit circle. See figure 2–11. Observe that the absolute values of the x- and y-coordinates of any point not on an axis describe the lengths of two sides of a right triangle with hypotenuse of length one. The Pythagorean theorem shows that for these points $x^2 + y^2 = 1$. Those points of the circle that are on an axis also satisfy this equation.

The circumference C of a circle with radius r is the distance around the circle. This distance is found using the relation $C = 2\pi r$. Since the radius r for the unit circle is one, its circumference is $C = 2\pi$ (about 6.28 units).

Note The constant π is approximately 3.14159. It is a much-used number, about which entire books have been written. It is an irrational number, and has been approximated to over a billion digits![3]

Radian measure

A second system of angle measurement is called **radians.** This system is used extensively in engineering and scientific applications, as well as in the calculus. We will use it throughout the rest of this book. To define this system of angle measurement we use the unit circle.

Let θ be an angle in standard position, and let s represent the distance from the point (1,0) along the circumference of the unit circle to the terminal

[3]An interesting book on π is *A History of* π by Petr Beckmann, Golem Press, Boulder, Colo., 1977. Gregory V. and David V. Chudnovsky of Columbia University calculated 1,011,196,691 digits of π in 1989.

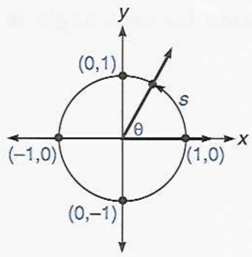


Figure 2–12

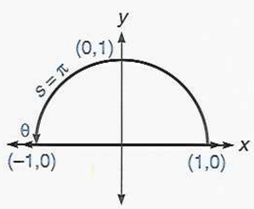


Figure 2–13

side of θ. The distance s is called the **arc length** (see figure 2–12). If the distance is measured in a clockwise direction we say s is positive, and if in a counterclockwise direction s is negative.

We define the radian measure of an angle to be this arc length s.

> **Radian measure of an angle in standard position**
> Let θ be an angle in standard position. Let s be the corresponding arc length on the unit circle. Let s be positive if measured in the counterclockwise direction, and negative if measured in the clockwise direction.
> Then s is the radian measure of the angle θ.

For example, an angle of degree measure 180° has an arc length that corresponds to half the circumference of the unit circle. Thus, the corresponding radian measure is half of the circumference, or one-half of 2π, which is π. Thus, the radian measure of an angle that corresponds to a rotation of one-half a circle, in the counterclockwise direction, is π (see figure 2–13).

Conversions between radian and degree measure

Since 360° corresponds to a full revolution, and the circumference of the unit circle (2π) also corresponds to a complete revolution about the unit circle, the following relation is true.

$$\frac{\text{arc length } (s)}{\text{circumference } (2\pi)} = \frac{\text{measure of angle in degrees}}{360^\circ}$$

If we multiply each member by 2 we obtain the same true statement, but with smaller denominators of π and 180°.

We use this proportion[4] to convert between degree and radian measure.

> **Radian/degree proportion**
> Let θ be an angle in standard position with degree measure θ° and radian measure s. Then,
> $$\frac{s}{\pi} = \frac{\theta^\circ}{180^\circ}$$

The radian measure of an angle is a real number, defined with no units in mind. We often add the word radians after such a measure, but this is not necessary where it is clear that the real number refers to the measure of an angle. Observe that in the radian/degree proportion the ratio of degrees to degrees is unitless also. For example, $\frac{90^\circ}{180^\circ}$ is the same as the unitless ratio $\frac{1}{2}$.

We can describe the measure of an angle in standard position by using degree measure or by stating the arc length to which the angle corresponds on the unit circle (its radian measure). The proportion above shows the relationship between these two systems.

[4] A proportion is a statement of equality between two ratios (fractions).

■ ***Example 2–5 A*** Compute the radian or degree measure, given the measure for each angle in degrees or radians.

1. 90°

$$\frac{s}{\pi} = \frac{\theta°}{180°} \quad \text{Radian/degree proportion}$$

$$\frac{s}{\pi} = \frac{90°}{180°} \quad \text{Replace } \theta° \text{ by } 90°$$

$$s = \frac{90(\pi)}{180} \quad \text{Multiply each member by } \pi\text{; drop the reference to degrees}$$

$$s = \frac{\pi}{2} \quad \text{Simplify the fraction}$$

Thus, 90° corresponds to $\frac{\pi}{2}$ (radians).

2. −210°

$$\frac{s}{\pi} = \frac{\theta°}{180°} \quad \text{Radian/degree proportion}$$

$$\frac{s}{\pi} = \frac{-210°}{180°} \quad \text{Replace } \theta° \text{ with } -210°$$

$$s = \frac{-210(\pi)}{180°} = -\frac{7\pi}{6}$$

Therefore, −210° corresponds to $-\frac{7\pi}{6}$.

3. $\frac{7\pi}{5}$

$$\frac{s}{\pi} = \frac{\theta°}{180°} \quad \text{Radian/degree proportion}$$

$$\frac{\frac{7\pi}{5}}{\pi} = \frac{\theta°}{180°} \quad \text{Replace } s \text{ by } \frac{7\pi}{5}$$

$$\frac{7\pi}{5} \cdot \frac{1}{\pi} = \frac{\theta°}{180°} \quad \text{Division by } \pi \text{ is the same as multiplication by } \frac{1}{\pi}$$

$$\frac{7}{5} \cdot 180° = \theta° \quad \text{Multiply each member by } 180°$$

$$252° = \theta°$$

$\frac{7\pi}{5}$ (radians) corresponds to 252°.

4. 1

Note that this means $s = 1$ radian.

$$\frac{s}{\pi} = \frac{\theta^\circ}{180^\circ} \qquad \text{Radian/degree proportion}$$

$$\frac{1}{\pi} = \frac{\theta^\circ}{180^\circ}$$

$$\frac{180^\circ}{\pi} = \theta^\circ \qquad \text{Multiply each member by } 180^\circ$$

$$\frac{180^\circ}{\pi} = \theta^\circ$$

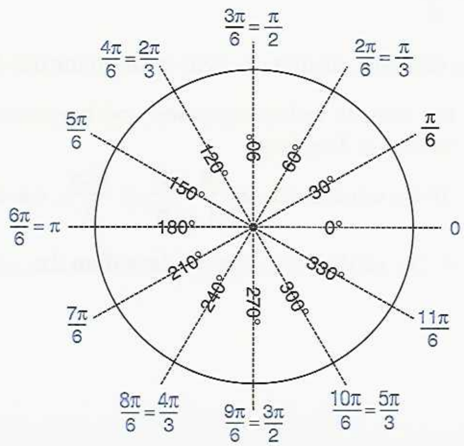
$$57.30^\circ \approx \theta^\circ \qquad \text{Decimal approximation to } \frac{180}{\pi}$$

Thus, 1 radian corresponds to $\frac{180^\circ}{\pi}$ or about 57.3°. ■

Note It is useful to remember that 1 radian is a little less than 60°, and that 2π radians exactly equals 360°.

Common radian measures

The unit circle can be very helpful in getting a feeling for radian measure. Those values of radian measure that correspond to quadrantal angles (0°, 90°, 180°, etc.) and to angles with reference angles of 30°, 45°, and 60° are common. In particular, the following correspondences are useful: $\frac{\pi}{6}$ and 30°, $\frac{\pi}{4}$ and 45°, and $\frac{\pi}{3}$ and 60°. The unit circle can be conveniently marked in terms of multiples of $\frac{\pi}{6}$ radians (30°) and of multiples of $\frac{\pi}{4}$ radians (45°). This is shown in figure 2–14.



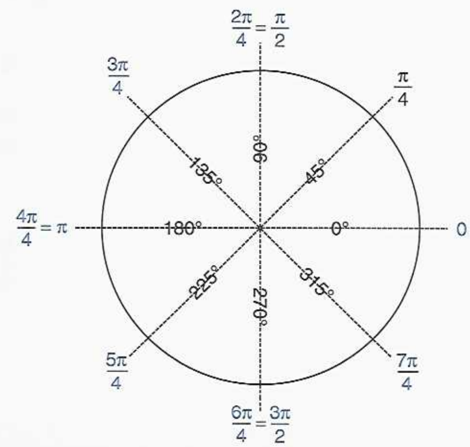


Figure 2–14

Love The Taste.
Taste The Love.

At Culver's® we can't think of anything better than serving up our creamy frozen custard and delicious classics cooked fresh the minute you order them. Which is why when we bring them to your table, they're always accompanied by a warm smile and a friendly offer to see if there's anything else we can get for you. So come on into your neighborhood Culver's and see for yourself. You might just be in love by the time you leave.

www.culvers.com

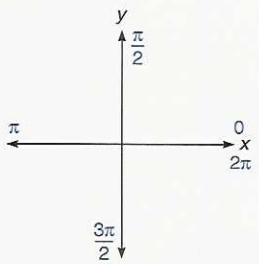


Figure 2–15

Coterminal and reference angles in radian measure

Figure 2–15 shows the smallest positive radian measure of the quadrantal angles, as well as the fact that a full revolution (circle) can be described by 2π radians. Observe that the quadrantal angles 0°, 90°, 180°, 270°, and 360° are, in radians, $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π.

In degree measure, all angles that differ in measure by integer multiples of 360° are coterminal. For radian measure the difference is multiples of 2π. If k is an integer (positive, zero, or negative), then integer multiples of 2π are $k \cdot 2\pi$, or $2k\pi$.

Coterminal angles, radian measure
Two angles α and β are coterminal if $\alpha = \beta + 2k\pi$, k an integer.

Reference angles are found in the same manner as with degree measure (section 2–2) except that 180° becomes π and 360° becomes 2π. If the measure of θ in radians is positive and less than 2π, the following rules give the value of θ', the reference angle.

Quadrant in which θ terminates	Value of θ', the reference angle
I	$\theta' = \theta$
II	$\theta' = \pi - \theta$
III	$\theta' = \theta - \pi$
IV	$\theta' = 2\pi - \theta$

■ ***Example 2–5 B***

For each angle θ find the least positive coterminal angle α (that is, $0 \le \alpha < 2\pi$) and then find the reference angle θ'.

1. $\theta = \frac{7\pi}{6}$

It is difficult, at first, to deal with quantities like $\frac{7\pi}{6}$. This is because we are not used to radian measure, and because it can be difficult to compare the values of fractions.

If we convert 2π to $\frac{2\pi}{1} \cdot \frac{6}{6} = \frac{12\pi}{6}$, we can see that $\frac{7\pi}{6} < \frac{12\pi}{6}$, or $\frac{7\pi}{6} < 2\pi$, so that θ is already less than 2π, so $\alpha = \theta = \frac{7\pi}{6}$.

To determine the reference angle for θ we must determine in which quadrant $\frac{7\pi}{6}$ terminates. If you cannot see that it terminates in quadrant III then use the following method.

Rewrite the quadrantal angles in terms of denominators of 6, the denominator of $\frac{7\pi}{6}$:

Quadrant		I		II		III		IV	
Quadrantal Angle	0		$\frac{\pi}{2}$		π		$\frac{3\pi}{2}$		2π
Denominator of 6	$\frac{0}{6}$		$\frac{3\pi}{6}$		$\frac{6\pi}{6}$		$\frac{9\pi}{6}$		$\frac{12\pi}{6}$

Now observe that $\frac{6\pi}{6} < \frac{7\pi}{6} < \frac{9\pi}{6}$ or $\pi < \frac{7\pi}{6} < \frac{3\pi}{2}$, so $\frac{7\pi}{6}$ is in quadrant III.

$$\theta' = \frac{7\pi}{6} - \pi = \frac{7\pi}{6} - \frac{6\pi}{6} = \frac{\pi}{6}$$

In quadrant III, $\theta' = \theta - \pi$

Thus, the reference angle for $\frac{7\pi}{6}$ is $\frac{\pi}{6}$.

2. $\theta = \frac{11\pi}{3}$

Since $2\pi = \frac{6\pi}{3}$, we see that $\theta > 2\pi$. We must subtract multiples of 2π until we arrive at an angle α, $0 \le \alpha < 2\pi$.

$$\frac{11\pi}{3} - 2\pi = \frac{11\pi}{3} - \frac{6\pi}{3} = \frac{5\pi}{3}$$

Since $\frac{5\pi}{3} < 2\pi$, the angle α is $\frac{5\pi}{3}$. Thus, $\frac{11\pi}{3}$ is coterminal with $\frac{5\pi}{3}$. To locate in which quadrant the angle $\frac{5\pi}{3}$ terminates, rewrite $\frac{5\pi}{3}$ as $\frac{10\pi}{6}$ and the quadrantal angles in terms of denominators of 6. (It is not convenient to rewrite the quadrantal angles in terms of denominators of 3.)

Examining the values used in part 1 of this example we see that $\frac{9\pi}{6} < \frac{10\pi}{6} < \frac{12\pi}{6}$ so $\frac{3\pi}{2} < \frac{5\pi}{3} < 2\pi$ and $\frac{5\pi}{3}$ is in quadrant IV. Thus,

$$\theta' = 2\pi - \frac{5\pi}{3} = \frac{6\pi}{3} - \frac{5\pi}{3} = \frac{\pi}{3}.$$

3. $\theta = -\frac{13\pi}{4}$

We first add multiples of $2\pi = \frac{8\pi}{4}$ to obtain a positive valued coterminal angle α. It is clear that one multiple of $\frac{8\pi}{4}$ will not give a positive result, so we will add two multiples.

$$-\frac{13\pi}{4} + 2\left(\frac{8\pi}{4}\right) = -\frac{13\pi}{4} + \frac{16\pi}{4} = \frac{3\pi}{4} = \alpha$$

To find a reference angle we will use $\frac{3\pi}{4}$ instead of $-\frac{13\pi}{4}$. We rewrite the quadrantal angles in terms of denominators of 4.

Quadrant		I		II		III		IV	
Quadrantal Angle	0		$\frac{\pi}{2}$		π		$\frac{3\pi}{2}$		2π
Denominator of 6	$\frac{0}{4}$		$\frac{2\pi}{4}$		$\frac{4\pi}{4}$		$\frac{6\pi}{4}$		$\frac{8\pi}{4}$

Since $\frac{2\pi}{4} < \frac{3\pi}{4} < \frac{4\pi}{4}$, we see that $\frac{\pi}{2} < \frac{3\pi}{4}\pi$, so $\theta = \frac{3\pi}{4}$ is in quadrant II, and $\theta' = \pi - \frac{3\pi}{4} = \frac{4\pi}{4} - \frac{3\pi}{4} = \frac{\pi}{4}$.

Radian measure and arc length in any circle

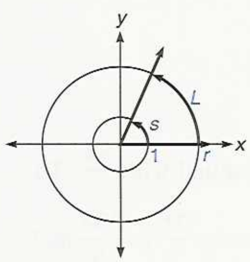


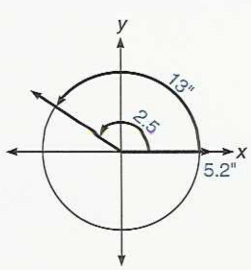
Figure 2–16

A simple relation exists between the radian measure s of an angle θ and arc length L determined by that angle on the circumference of any circle (figure 2–16). Geometry tells us that corresponding parts of similar figures form equal ratios. This means, in this case, that $\frac{s}{1} = \frac{L}{r}$, or $L = rs$. Thus, *if s is the radian measure of an angle with vertex at the center of a circle of radius r, and L is the corresponding arc length,* then

$$L = rs$$

Thus, the arc length L on any circle equals the product of the radius of the circle and the radian measure of the related angle.

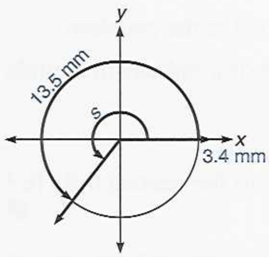
Example 2–5 C



Use the relation $L = rs$ to solve each problem.

1. Find the length of the arc determined by a central angle of measure 2.5 radians on a circle of radius 5.2 inches.

$$L = rs$$
$$L = 5.2(2.5) \quad \text{Replace } r \text{ with 5.2, } s \text{ with 2.5}$$
$$= 13 \text{ inches}$$



2. Find the measure in radians of the central angle corresponding to an arc length of 13.5 mm on a circle of diameter 6.8 mm.

$r = 3.4$ mm — One-half the diameter

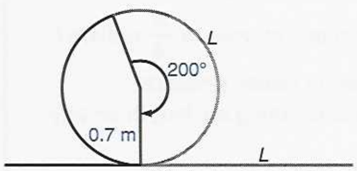
$L = rs$

$13.5 \text{ mm} = (3.4 \text{ mm})s$ — Substitute known values

$$\frac{13.5}{3.4} = s$$

$3.97 \approx s$ — Rounding to nearest 0.1

Thus, the central angle measures 3.97 radians.



3. A railroad car has wheels with diameter 1.4 m (meters). If the wheels move through an angle of 200°, how far does the train move?

As illustrated, the distance the train will move is the same as the arc length L on the wheel. This length is determined by the central angle of 200°. We will find the measure of the central angle, θ, in radians, then use the relation $L = rs$.

$$\frac{200°}{180°} = \frac{s}{\pi} \qquad \frac{\theta°}{180°} = \frac{s}{\pi}$$

$$\frac{10}{9} = \frac{s}{\pi} \qquad \text{Reduce}$$

$$\frac{10\pi}{9} = s \qquad \text{Multiply each member by } \pi$$

$$L = rs$$

$$L = 0.7\left(\frac{10\pi}{9}\right) \qquad \text{The radius } r \text{ is half the diameter of 1.4 m; } s = \theta \text{ (in radians)}$$

$$L \approx 2.4 \text{ m}$$

Thus, the train moves 2.4 meters when the wheels move through an angle of 200°.

Area of a sector of a circle

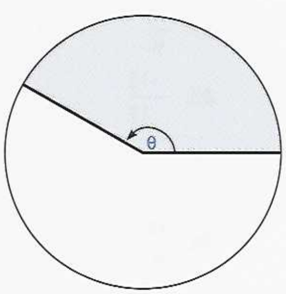


Figure 2–17

A sector of a circle is that part enclosed between two radii. See the shaded part of figure 2–17 for an example. The area A of a circle with radius r is determined by the equation $A = \pi r^2$. The area of a sector of a circle is proportional to the measure of the central angle θ determining the sector. Thus, the area of a sector with central angle θ with measure s radians is part of a circle determined by

angle × area of whole circle

$$\frac{s}{2\pi} \times \pi r^2 = \frac{s}{2} r^2$$

Thus, the area of a sector of a circle of radius r is

$$A_s = \frac{s}{2} r^2$$

■ ***Example 2–5 D***

Use the formula for the area of a sector of a circle to solve the problem.

Find the area of a sector determined by a central angle of 2 radians in a circle of diameter 8.5″.

The radius r is half the diameter, or 4.25″.

$A_s = \frac{s}{2}r^2 = \frac{2}{2}(4.25^2) \approx 18.06$ square inches, rounded to the nearest 0.01 in.^{2} ■

Mastery points

Can you

- State the equation of the unit circle?
- Convert between degree and radian measure for angles?
- Mark off a unit circle in units of $\frac{\pi}{6}$ radians and in units of $\frac{\pi}{4}$ radians?
- Find the reference angle for an angle given in radian measure?
- Use the relation $L = rs$ to solve problems concerning arc length on any circle?
- Use the relation $A_s = \frac{s}{2}r^2$ to find the area of a sector of a circle?

Exercise 2–5

1. State the algebraic relation (equation) that describes the unit circle.

Convert the following degree measures to radian measures. Leave your answers both in exact form and approximated to two decimal places.

2. 30° **3.** 45° **4.** 60° **5.** 100° **6.** −200° **7.** −300°
8. −135° **9.** 270° **10.** 750° **11.** 127° **12.** −422° **13.** −305°

Convert the following radian measures into degree measures. Leave answers both in exact form and approximated to two decimal places.

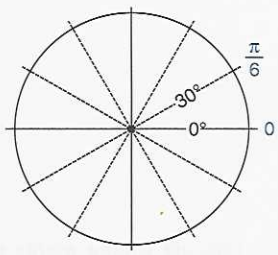
14. $\frac{5\pi}{2}$ **15.** $\frac{11\pi}{6}$ **16.** $\frac{2\pi}{7}$ **17.** $\frac{3\pi}{5}$ **18.** $\frac{10\pi}{9}$ **19.** $\frac{2\pi}{9}$
20. $-\frac{5\pi}{3}$ **21.** $-\frac{17\pi}{6}$ **22.** $-\frac{5\pi}{7}$ **23.** $\frac{3}{2}$ **24.** $\frac{11}{6}$ **25.** $-\frac{12}{17}$
26. 1.5 **27.** 2 **28.** 3.25 **29.** −5 **30.** −6

Find the reference angle for the following angles.

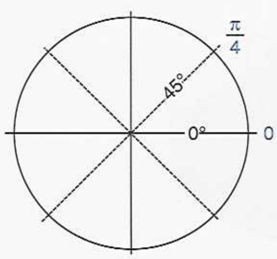
31. $\frac{2\pi}{3}$ **32.** $\frac{5\pi}{4}$ **33.** $\frac{11\pi}{6}$ **34.** $\frac{7\pi}{6}$ **35.** $\frac{4\pi}{3}$ **36.** $\frac{5\pi}{6}$
37. $\frac{3\pi}{4}$ **38.** $\frac{7\pi}{4}$ **39.** $\frac{5\pi}{3}$ **40.** $-\frac{\pi}{4}$ **41.** $-\frac{2\pi}{3}$ **42.** $-\frac{5\pi}{3}$
43. $-\frac{7\pi}{6}$ **44.** $-\frac{\pi}{6}$

Solve the following problems.

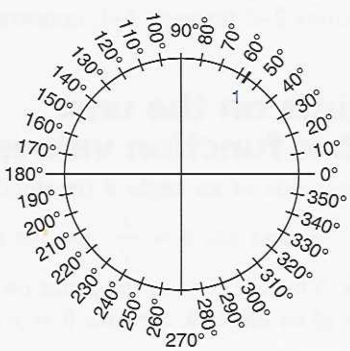
45. The circle is marked off in units of $\frac{\pi}{6}$ (or 30°). Mark each angle shown with its appropriate measure. Reduce fractions.



46. The circle is marked off in units of $\frac{\pi}{4}$ (or 45°). Mark each angle shown with its appropriate measure. Reduce fractions.



47. The circle is marked off in degrees. Also shown is the approximate location of 1 radian, which is approximately 57°. Mark off the approximate locations of 2, 3, 4, 5, and 6 radians.



48. Find the length of the arc determined by a central angle of 2.1 (radians) on a circle of diameter 10 inches.

49. Find the measure, in radians, of a central angle on a circle of radius 4.5 mm (millimeter) determined by an arc length of 12 mm, to the nearest 0.1 radian.

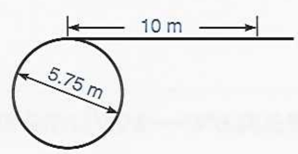
50. Find the length of the arc determined by a central angle of 300° on a circle of diameter 12 mm.

51. Find the length of the arc determined by a central angle of 45° on a circle of radius 8.3 inches, to the nearest 0.1 inch.

52. Find the measure, in both radians and degrees, of the central angle determined by an arc length of 14.5 mm on a circle with diameter 10.3 mm. Round both answers to the nearest tenth.

53. The diameter of a wheel on an automobile is 32.4 inches. If the wheel moves through an angle of 85°, how far will the car move?

54. The diameter of a wheel that moves the cable of a ski lift is 5.75 meters, as shown in the diagram. Through what angle, in degrees, does the wheel have to move to advance one of the chairs a distance of 10 meters?



Find the area of the sector of a circle determined by the given angle and radius. Where necessary round the answer to two decimal places.

55. 4, 7 inches

56. $\frac{\pi}{3}$, 10 millimeters

57. $\frac{3\pi}{5}$, 6 centimeters

58. 2.4, 5 inches

59. $\frac{5}{12}$, 6 inches

60. $\frac{6\pi}{5}$, 22 centimeters

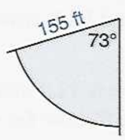
61. 15°, 9 millimeters

62. 135°, 24 inches

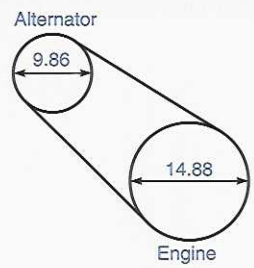
63. Find the radian measure of the central angle necessary to form a sector of area 14.6 cm^2 on a circle of radius 4.85 centimeters. Round the answer to two decimal places.

64. Find the *degree* measure of the central angle necessary to form a sector of area 200 in.^2 on a circle of diameter 50 inches. Round to the nearest 0.1°.

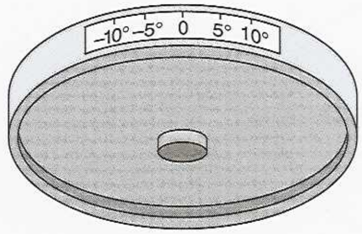
65. The figure shows a sector of a circle that was painted on the concrete in front of an airport terminal. The radius is now to be extended by 25 feet, to a total of 180 feet. The paint that will cover the unpainted area in the new, larger sector covers about 150 square feet per gallon. How many gallons will it take, to the nearest half gallon, to cover the new, unpainted area?



66. The alternator on an automobile engine is attached by a belt to a wheel on the engine. The wheel on the engine has a diameter of 14.88 cm (see the figure), and the wheel on the alternator has a diameter of 9.86 cm. If the wheel driven by the engine moves through an angle of 2.85 radians, through what angle does the alternator move, to the nearest 0.01 radian?



67. A decal is being made to indicate timing marks on a wheel attached to the front of an engine (see the diagram). The radius of the wheel is 86.6 mm. What should the distance be between the $-10°$ and $10°$ marks, to the nearest millimeter?



2–6 Radian measure—values of the trigonometric functions

In this section we relate radian measure to the trigonometric functions. The concepts here are the same as seen in sections 2–2 through 2–4, concerning degree measure.

Relationship between points on the unit circle and the trigonometric function values

Recall that if (x,y) is a point on the terminal side of an angle θ (in standard position), and $r = \sqrt{x^2 + y^2}$, then $\sin\theta = \dfrac{y}{r}$ and $\cos\theta = \dfrac{x}{r}$. On the unit circle, $r = 1$, so $\sin\theta = y$, and $\cos\theta = x$. Thus, *if (x,y) is the point on the unit circle that intersects the terminal side of an angle θ, then $\sin\theta = y$ and $\cos\theta = x$.* Figure 2–18 shows this fact.

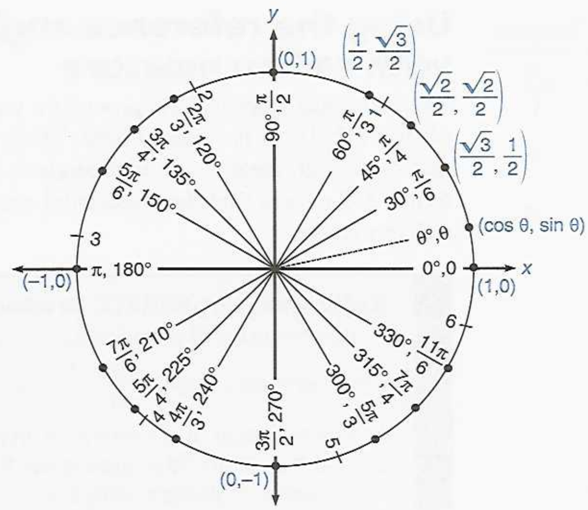


Figure 2–18

This figure is very useful because it shows the degree and radian measure for many common angles with measure between 0° and 360° (0 and 2π in radian measure). The angles shown are either quadrantal or have reference angles of measure 30° $\left(\frac{\pi}{6}\right)$, 45° $\left(\frac{\pi}{4}\right)$, or 60° $\left(\frac{\pi}{3}\right)$. The figure also shows the point on the terminal side of an angle where it meets the unit circle. As stated above, the (x,y) pair at each point on the unit circle is $(\cos\theta, \sin\theta)$ for the corresponding angle. Note that the radian measure is shown for the values $1, 2, 3, \ldots, 6$ as well as for the multiples of π mentioned above. For example, 2 (radians) is near $\frac{2\pi}{3} \approx 2.1$ (radians), or 120°.

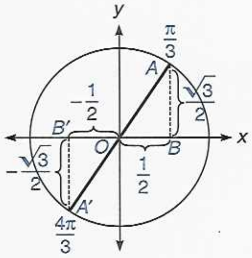


Figure 2–19

Observe that you can find the sine or cosine value for any of the angles shown by observing the symmetries in figure 2–18. For example, the coordinates at $\frac{4\pi}{3}$ must be $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. As seen in figure 2–19, traveling through the origin from one point on the unit circle to another simply changes the signs of the x- and y-coordinates. Thus, $\cos\frac{4\pi}{3}$ is $-\frac{1}{2}$ and $\sin\frac{4\pi}{3}$ is $-\frac{\sqrt{3}}{2}$.

Similarly, the coordinates at $\frac{5\pi}{6}$ (figure 2–18) must have the same y-coordinate as at $\frac{\pi}{6}$, but the opposite of the x-coordinate. Thus, the coordinates there must be $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, and from this point we know that $\sin\frac{5\pi}{6} = \sin 150° = y = \frac{1}{2}$, $\cos\frac{5\pi}{6} = \cos 150° = -\frac{\sqrt{3}}{2}$.

	Sine	Cosine	Tangent
$\frac{\pi}{6}$, 30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$, 45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$, 60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Table 2–2

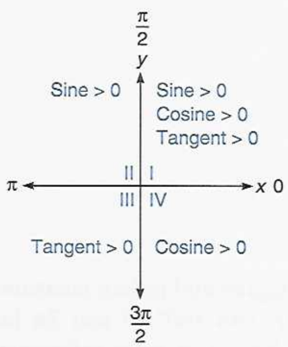


Figure 2–20

Using the reference angle/ASTC procedure with radian measure

The reference angle/ASTC procedure (section 2–3) can also be used instead of figure 2–18. It is restated here. Table 2–2 is the same as table 2–1 except that the radian measure of each angle is included. Figure 2–20 is the same as figure 2–5 except that the quadrantal angles less than 2π (360°) are shown in radian measure.

> **Reference angle/ASTC procedure for radian measure**
>
> To find the value of a trigonometric function for a nonquadrantal angle whose reference angle is $\frac{\pi}{6}$, $\frac{\pi}{4}$, or $\frac{\pi}{3}$:
>
> 1. Find the value of the reference angle.
> 2. Find the value of the appropriate trigonometric function for the reference angle from table 2–2.
> 3. Determine the sign of this value using the ASTC rule (figure 2–20).

■ *Example 2–6 A*

Use table 2–2 and the ASTC rule to find the exact value of each expression.

1. $\cos \frac{7\pi}{6}$

Step 1: $\theta' = \frac{\pi}{6}$ — This was shown in part 1 of example 2–5 B

Step 2: $\cos \theta' = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$ — Table 2–2

Step 3: $\cos \frac{7\pi}{6} < 0$ — ASTC rule; $\cos \theta < 0$ in quadrant III

Thus, $\cos \frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$.

2. $\sin \frac{11\pi}{3}$

Step 1: $\theta' = \frac{\pi}{3}$ — Shown in part 2 of example 2–5 B

Step 2: $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ — Table 2–2

Step 3: $\sin \frac{5\pi}{3} < 0$ — ASTC rule; $\sin \theta < 0$ in quadrant IV

Thus, $\sin \frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$ and therefore $\sin \frac{11\pi}{3} = -\frac{\sqrt{3}}{2}$.

3. $\cot\left(-\frac{13\pi}{4}\right)$

$$\cot\left(-\frac{13\pi}{4}\right) = \frac{1}{\tan\left(-\frac{13\pi}{4}\right)}$$

In part 3 of example 2–5 B, we saw that $-\frac{13\pi}{4}$ and $\frac{3\pi}{4}$ are coterminal, and the reference angle for either is $\frac{\pi}{4}$, so

$$\tan\left(-\frac{13\pi}{4}\right) = \tan\frac{3\pi}{4}$$

We proceed to find $\tan\frac{3\pi}{4}$:

Step 1: The reference angle for $\frac{3\pi}{4}$ is $\frac{\pi}{4}$.

Step 2: $\tan\frac{\pi}{4} = 1$

Step 3: $\frac{3\pi}{4}$ terminates in quadrant II, where the tangent function is negative, so $\tan\frac{3\pi}{4} < 0$.

Thus, $\tan\frac{3\pi}{4} = -1$.

Now we can finish the problem.

$$\cot\left(-\frac{13\pi}{4}\right) = \frac{1}{\tan\left(-\frac{13\pi}{4}\right)} = \frac{1}{-1} = -1$$ ■

Calculators are programmed to accept angle input in radian measure. All scientific calculators have a key, often marked $\boxed{\text{DRG}}$ or $\boxed{\text{MODE}}$ to tell the calculator to accept angles in radian measure. On the TI-81 use the $\boxed{\text{MODE}}$ key to select the mode display. Then use the four cursor keys to darken Rad, and use $\boxed{\text{ENTER}}$ to change to radian mode. Use $\boxed{\text{QUIT}}$ ($\boxed{\text{2nd}}$ $\boxed{\text{CLEAR}}$) to exit the mode display.

Thus, for angles that are not coterminal with those in figure 2–18 we use the calculator, in radian mode. This is illustrated in example 2–6 B.

■ ***Example 2–6 B***

Find the required value with a calculator; round the answer to four decimal places.

Make sure the calculator is in radian mode.

1. sin 1.2

1.2 $\boxed{\sin}$ Display $\boxed{0.932039086}$

$\boxed{\text{TI-81}}$ $\boxed{\text{SIN}}$ 1.2 $\boxed{\text{ENTER}}$

$\sin 1.2 \approx 0.9320$

2. $\cot(-0.7)$

$$\cot(-0.7) = \frac{1}{\tan(-0.7)}$$

.7 $\boxed{+/-}$ $\boxed{\tan}$ $\boxed{1/x}$ Display $\boxed{-1.187241832}$

$\boxed{\text{TI-81}}$ $\boxed{(}$ $\boxed{\text{TAN}}$ $\boxed{(-)}$ .7 $\boxed{)}$ $\boxed{x^{-1}}$ $\boxed{\text{ENTER}}$

$\cot(-0.7) \approx -1.1872$ ■

As with degrees, we need to be able to find an angle in radians when given the value of one of the trigonometric functions for that angle, and information about the quadrant. As in section 2–3 we will restrict ourselves to the least (smallest value) nonnegative solution to trigonometric equations.

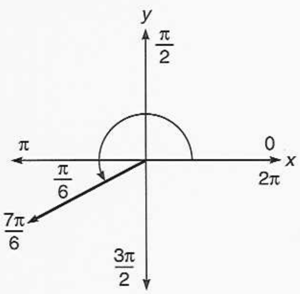
Since a reference angle is acute (between 0 and $\frac{\pi}{2}$ radians), and since the values of all the trigonometric functions are positive for acute angles (quadrant I, ASTC rule) we *always use a nonnegative value to find a reference angle.* This is illustrated in the following example.

In section 2–5 we saw formulas that give θ' if $0 < \theta < 2\pi$. If these are solved for θ, we obtain formulas for finding θ given θ' and the quadrant in which θ terminates.

Relationship between θ and θ' if $0 < \theta < 2\pi$

θ in Quadrant I:	$\theta' = \theta$	$\theta = \theta'$
θ in Quadrant II:	$\theta' = \pi - \theta$	$\theta = \pi - \theta'$
θ in Quadrant III:	$\theta' = \theta - \pi$	$\theta = \pi + \theta'$
θ in Quadrant IV:	$\theta' = 2\pi - \theta$	$\theta = 2\pi - \theta'$

■ *Example 2–6 C*



Find the least nonnegative value of θ, in radians. Round to two decimal places if necessary.

1. $\sin\theta = -\frac{1}{2}, \cos\theta < 0$

$\theta' = \sin^{-1}\frac{1}{2}$ — Always use a nonnegative value to find a reference angle

$\theta' = \frac{\pi}{6}$ — Table 2–2

θ is in quadrant III, since $\sin\theta < 0$ and $\cos\theta < 0$.

$\theta = \pi + \frac{\pi}{6}$ — In quadrant III $\theta = \pi + \theta'$

$\theta = \frac{7\pi}{6}$ — $\pi + \frac{\pi}{6} = \frac{6\pi}{6} + \frac{\pi}{6}$

2. $\cos\theta = -0.8, \tan\theta < 0$

$\theta' = \cos^{-1} 0.8$ — Use the nonnegative value

$\theta' \approx 0.64$ radians — 0.8 $\boxed{\cos^{-1}}$ (In radian mode.)

$\boxed{0.6435011088}$

$\boxed{\text{TI-81}}$ $\boxed{\text{COS}^{-1}}$ .8 $\boxed{\text{ENTER}}$

θ is in quadrant II since $\cos\theta < 0$, $\tan\theta < 0$.

$$\theta = \pi - \theta' \approx 2.50 \text{ radians}$$

Solutions to trigonometric equations

In sections 1–4 and 2–3 we examined solutions to equations involving the trigonometric ratios and functions in terms of degrees. Chapter 5 treats this subject in more depth, but this is a very important topic, so we revisit it here.

In the problems in example 2–6 D there are an unlimited number of answers. We will look for one basic answer in each case. By limiting the problems to practically all nonnegative values we can almost always find answers in the first quadrant in a very straightforward way.

The objective here is not to completely solve these equations, but to get used to the algebra involved in solving trigonometric equations. These equations use the following facts, which are presented in depth in chapters 4 and 5:

if $\sin\theta = k$, then one solution is $\theta = \sin^{-1}k$
if $\cos\theta = k$, then one solution is $\theta = \cos^{-1}k$
if $\tan\theta = k$, then one solution is $\theta = \tan^{-1}k$

We will see some equations that use the **zero product property:** If a product is zero, then at least one factor is zero. For example, if

$$\sin x(\sin x - 1) = 0$$

then either

$$\sin x = 0$$

or

$$\sin x - 1 = 0$$

■ ***Example 2–6 D***

Find a solution to each equation, in radians. Round to two decimal places when necessary.

1. $4 \cos x = 1$

$$4 \cos x = 1$$
$$\cos x = \tfrac{1}{4}$$
$$x = \cos^{-1}\tfrac{1}{4} \approx 1.32 \text{ (radians)}$$

2. $3 \sin 2x = 2$

$$3 \sin 2x = 2$$
$$\sin 2x = \tfrac{2}{3} \qquad \text{Divide both members by 3}$$
$$2x = \sin^{-1}\tfrac{2}{3}$$
$$x = \tfrac{1}{2}\sin^{-1}\tfrac{2}{3}$$
$$x \approx 0.36 \text{ (radians)}$$

1 [÷] 2 [×] [(] 2 [÷] 3 [)] [sin⁻¹] [=]

[TI-81] [(] 1 [÷] 2 [)] [SIN⁻¹] [(] 2 [÷] 3 [)] [ENTER]

3. $\tan\frac{x}{2} = 1.5$

$$\tan\frac{x}{2} = 1.5$$
$$\frac{x}{2} = \tan^{-1} 1.5$$
$$x = 2 \tan^{-1} 1.5$$
$$x \approx 1.97 \text{ (radians)}$$

4. $(2 \sin \theta - 1)(\sin \theta + 1) = 0$

This is a product that equals zero, so we apply the zero product property.

$$2 \sin \theta - 1 = 0 \quad \text{or} \quad \sin \theta + 1 = 0$$
$$2 \sin \theta = 1 \quad \text{or} \quad \sin \theta = -1$$
$$\sin \theta = \tfrac{1}{2} \qquad \theta = \frac{3\pi}{2}$$
$$\theta = \sin^{-1}\tfrac{1}{2}$$
$$\theta = \frac{\pi}{6}$$

Thus, there are two solutions, $\frac{\pi}{6}$ and $\frac{3\pi}{2}$. ■

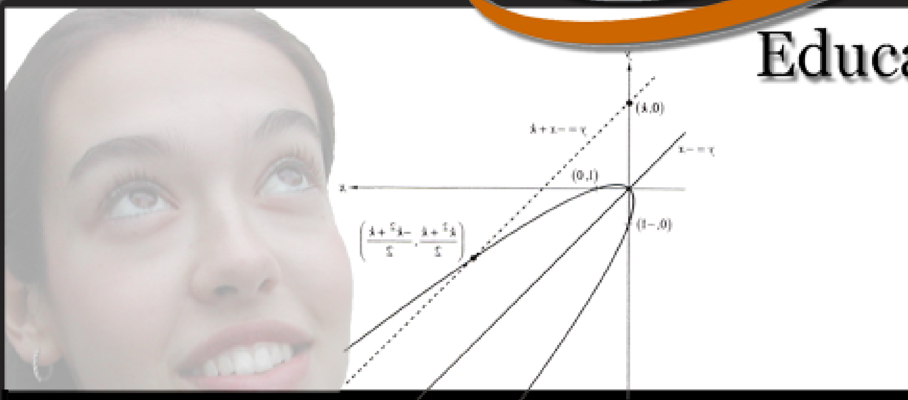
Mastery points

Can you

- Find the exact value of a trigonometric function for an angle whose reference angle is $\frac{\pi}{6}$, $\frac{\pi}{4}$, or $\frac{\pi}{3}$, as well as quadrantal angles?
- Find approximate values of the trigonometric functions with a calculator when the angle is given in radians?
- Find a solution, in radians, to certain trigonometric equations?

GetMath®.com

Educational Software



We help students get math. It's just that simple.

If you are afraid of math, think you cannot do math, go blank on math tests, our 5.0 learning system will help you get math.

Audio/Video Lectures On Every Page – No other learning system is enhanced with audio/video lectures embedded throughout the material. As you are studying, there are video lectures next to the material explaining the concepts, procedures, and examples as they appear. Within the exercise sets and chapter review material, you will find video lectures next to the appropriate problems, which reinforce and solidify the learning experience.

Step-by-step solutions to all problems – Instead of just an answer, every problem has a step-by-step solution, which allows you to verify that you are doing the problems correctly. Should you make an error, there is no wondering, “where did I go wrong”, the step-by-step solution will pinpoint the step in which the error occurred.

GetMath Educational Software is your link to success in mathematics – Wherever you are in the material, you are just a click away from all of the support features of a learning system that has been developed and fine-tuned over the last two decades.

GetMath.com Educational Software has joined forces with Freeload Press in helping liberate the textbook so students from all financial backgrounds can use these important learning tools. GetMath is bringing more than 30 textbooks and ancillaries to the FreeLoad site for free download. Several million students have purchased at least one of our textbooks, but you will now be able to start using those same textbooks along with the resource material for free.

Your path to success is made easy – with video lectures, quick reference examples, notes to the student, quick check problems, concept boxes, trial problems, real-life applications, mastery points, core exercises, procedure boxes, review exercises, explore-to-learn problems, functional use of color as a guide, and over 20 years of in-the- classroom refinements.

Exercise 2–6

Find the exact function values for the following angles.

1. $\sin \frac{2\pi}{3}$ **2.** $\tan \frac{5\pi}{4}$ **3.** $\cos \frac{11\pi}{6}$ **4.** $\tan \frac{7\pi}{6}$ **5.** $\cos \frac{4\pi}{3}$

6. $\sin \frac{5\pi}{6}$ **7.** $\sin \frac{3\pi}{4}$ **8.** $\cos \frac{7\pi}{4}$ **9.** $\tan \frac{5\pi}{3}$ **10.** $\sin\left(-\frac{\pi}{4}\right)$

11. $\tan\left(-\frac{2\pi}{3}\right)$ **12.** $\sec(-\pi)$ **13.** $\sin\left(-\frac{7\pi}{6}\right)$ **14.** $\cos\left(-\frac{\pi}{6}\right)$

Find the following function values where the angle is given in radian measure. Round your answers to four decimal places.

15. sin 0.9 **16.** cos 1.1 **17.** tan 0.5 **18.** sec 1.4 **19.** csc 0.7 **20.** cot 1.5
21. sin 2.3 **22.** cos 3.5 **23.** tan 4.1 **24.** sec 5.2 **25.** csc 2.5 **26.** cot 1.9

Find the least nonnegative value of θ, in radians. Round to two decimal places if necessary.

27. $\cos\theta = -\frac{1}{2}$, $\tan\theta > 0$ **28.** $\tan\theta = \sqrt{3}$, $\sin\theta < 0$ **29.** $\sin\theta = -\frac{\sqrt{3}}{2}$, $\tan\theta < 0$

30. $\sec\theta = \frac{2}{\sqrt{3}}$, $\sin\theta > 0$ **31.** $\cot\theta = -1$, $\sec\theta > 0$ **32.** $\tan\theta = -\sqrt{3}$, $\cos\theta > 0$

33. $\csc\theta = -2$, $\cos\theta > 0$ **34.** $\sin\theta = -0.5624$, $\tan\theta > 0$ **35.** $\tan\theta = -2.5$, $\csc\theta > 0$
36. $\cot\theta = -0.3$, $\sin\theta > 0$ **37.** $\cos\theta = -0.885$, $\tan\theta > 0$ **38.** $\sin\theta = -0.2258$, $\cos\theta > 0$
39. $\csc\theta = -3$, $\sec\theta < 0$

Find one solution to the following equations, in radians. Round to two decimal places if necessary.

40. $4\cos 3\theta = 2$ **41.** $3\sin 2\theta = 1$ **42.** $\frac{1}{3}\tan 2\theta = 1$ **43.** $2\tan\theta = 5$

44. $2\sin 3\theta = \sqrt{3}$ **45.** $\sin\frac{\theta}{2} = 1$ **46.** $2\sec 3\theta = 6$ **47.** $\sin 3\theta = \frac{1}{2}$

48. $(2\sin\theta - 1)(\sin\theta - 1) = 0$ **49.** $\cos\theta(2\cos\theta - 1) = 0$
50. $\tan\theta(\tan\theta - 1) = 0$ **51.** $(2\sin\theta - \sqrt{3})(\sin\theta - 1) = 0$

52. In a certain series circuit the applied voltage V in volts is determined by the function

$$V = 200\sin(35t + 1)$$

where t represents time in milliseconds and the expression $35t + 1$ is in radians. Compute V to the nearest 0.1 volt for the following values of t:

a. 0 **b.** 0.1 **c.** 0.8 **d.** 1

53. The position d at the end of a spring, under certain initial conditions, as a function of time t in seconds, is

$$d = \frac{1}{3}\cos 8t - \frac{1}{4}\sin 8t$$

Compute d for the following values of t:

a. $\frac{1}{8}$ **b.** $\frac{1}{4}$

54. An equation that arises in finding the trajectory of a rocket is

$$r = \frac{p}{1 + e\cos(s - C)}$$

Find r if $p = 200$, $e = 1.5$, $C = 0.5$, and

a. $s = 1$ **b.** $s = 1.25$

55. In interpreting an electrocardiogram a cardiologist obtains values a and b from the heights of certain peaks, and the value of angle θ, which depends on where the electrodes are attached to the patient. Then the value V is calculated from the expression

$$V = \frac{\sqrt{a^2 + b^2 - 2ab\cos\theta}}{\sin\theta}.$$

The value of V helps the cardiologist diagnose specific heart abnormalities. Compute V if $a = 6.2$ cm, $b = 3.5$ cm, and $\theta = 2.6$ (radians). Round the answer to two decimal places. The units will be centimeters.

56. If a 100-pound force pushes on an object at an angle θ that is measured between the direction of the force and the direction of motion of the object, then the amount of force actually moving the object is given by the function

$$f(\theta) = 100\cos\theta$$

Find the force f for the following values of θ (all in radians), to one decimal place:

a. 0.2 **b.** 0.4 **c.** 0.6 **d.** 1

57. An equation that can be used to compute sin x, if x is in radians, is called the *Maclaurin series* for the sine function. It is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots,$$

where

$$3! = 1 \cdot 2 \cdot 3 = 6,$$
$$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120,$$
$$7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 5{,}040, \text{ etc.}$$

($n!$ is read "n factorial" and is defined as $1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot n$.)

Although the Maclaurin series goes on forever, good accuracy is obtained by using the first few terms. Use the first four of the five terms shown here to compute approximations to

a. sin 0.1 **b.** sin 0.5 **c.** sin 1 **d.** $\sin \frac{\pi}{6}$

Check the results with the sine key of a calculator. (Some computers use a method similar to this for computing the trigonometric functions.)

58. (See problem 57.) The Maclaurin series for the cosine function, if x is in radians, is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots.$$

Use the first four of the five terms shown above to calculate approximations to

a. cos 0.8 **b.** cos 1 **c.** cos 1.3
d. Approximate cos 10° by first converting 10° to radians.

Chapter 2 summary

- A function is a set of ordered pairs having the property that no first element of the ordered pairs repeats.
- A function is one to one if no second element of the ordered pairs repeats.
- A one-to-one function f has an inverse function f^{-1}.
- **An angle in standard position** is formed by two rays, one of which always lies on the nonnegative portion of the x-axis. This ray is called the initial side. The second ray is called the terminal side. It may be in any quadrant or along any axis.
- Two angles α and β are said to be coterminal if $\alpha = \beta + k(360°)$, k an integer.
- **The trigonometric functions** Let θ be an angle in standard position, and let (x,y) be any point on the terminal side of the angle, except (0,0). Let $r = \sqrt{x^2 + y^2}$ be the distance from the origin to the point. Then

$$\sin \theta = \frac{y}{r}, \quad \cos \theta = \frac{x}{r}, \quad \tan \theta = \frac{y}{x}$$

$$\csc \theta = \frac{r}{y}, \quad \sec \theta = \frac{r}{x}, \quad \cot \theta = \frac{x}{y}$$

- **Tangent/cotangent identities**

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \ \cot \theta = \frac{\cos \theta}{\sin \theta}.$$

- **The ASTC rule**

In quadrant I,	All the trigonometric functions are positive.
In quadrant II, the	Sine function is positive.
In quadrant III, the	Tangent function is positive.
In quadrant IV, the	Cosine function is positive.

- Angles whose degree measures are integer multiples of 90°, such as 0°, ± 90°, ± 180°, ±270°, etc., are called quadrantal angles.
- A reference angle for a nonquadrantal angle is the acute angle formed by the terminal side of the angle and the x-axis.
- If $0° < \theta < 360°$, then the following relate the reference angle θ' and the angle θ.

θ in Quadrant I:	$\theta' = \theta$	$\theta = \theta'$
θ in Quadrant II:	$\theta' = 180° - \theta$	$\theta = 180° - \theta'$
θ in Quadrant III:	$\theta' = \theta - 180°$	$\theta = 180° + \theta'$
θ in Quadrant IV:	$\theta' = 360° - \theta$	$\theta = 360° - \theta'$

- The absolute value of a trigonometric function for any angle is the same as the trigonometric ratio for the corresponding reference angle.

- **Reference angle/ASTC procedure** To find the exact value of a trigonometric function for a nonquadrantal angle whose reference angle is $30°\left(\text{or } \frac{\pi}{6}\right)$, $45°\left(\text{or } \frac{\pi}{4}\right)$, or $60°\left(\text{or } \frac{\pi}{3}\right)$:
  1. Find the value of the reference angle.
  2. Find the value of the appropriate trigonometric ratio for the reference angle from the following table.
  3. Determine the sign of this value using the ASTC rule.

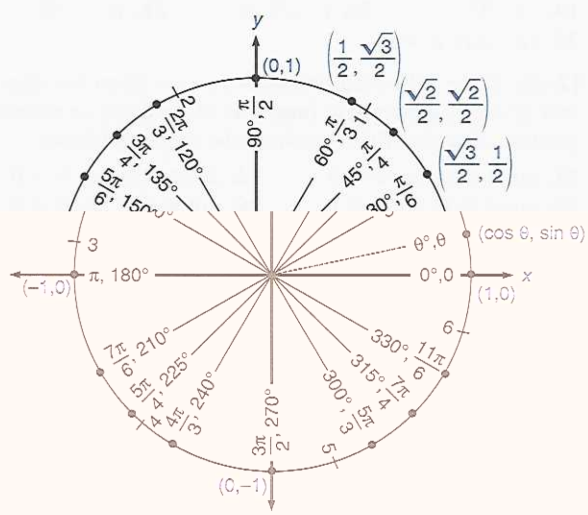
	Sine	Cosine	Tangent
$30°, \frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$45°, \frac{\pi}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°, \frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

- Solutions to simple trigonometric equations:
  if $\sin\theta = k$, then one solution for θ is $\theta = \sin^{-1} k$
  if $\cos\theta = k$, then one solution for θ is $\theta = \cos^{-1} k$
  if $\tan\theta = k$, then one solution for θ is $\theta = \tan^{-1} k$
- Finding the least nonnegative measure of an angle from a trigonometric function value and information about a quadrant.
  1. If necessary use the ASTC rule to determine the quadrant for the terminal side of the angle.
  2. Use $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$ to find θ'. Use the absolute value of the given trigonometric function value.
  3. Apply θ' to the correct quadrant to determine the value of θ.
- **A reference triangle** is a right triangle with one leg on the x-axis and one leg parallel to the y-axis.
- The circle with radius one and center at the origin is described by the equation $x^2 + y^2 = 1$. It is called the unit circle.
- Let θ be an angle in standard position. Let s be the corresponding arc length on the unit circle. Let s be positive if measured in the counterclockwise direction, and negative if measured in the clockwise direction. Then s is the radian measure of the angle θ.

- **Radian/degree proportion** Let θ be an angle in standard position with degree measure $\theta°$ and radian measure s. Then $\frac{s}{\pi} = \frac{\theta°}{180°}$.
- Relationship between an angle θ, $0 < \theta < 2\pi$, and θ', its reference angle in radian measure.

Quadrant in which θ terminates		
I	$\theta' = \theta$	$\theta = \theta'$
II	$\theta' = \pi - \theta$	$\theta = \pi - \theta'$
III	$\theta' = \theta - \pi$	$\theta = \pi + \theta'$
IV	$\theta' = 2\pi - \theta$	$\theta = 2\pi - \theta'$

- If s is the radian measure of an angle with vertex at the center of a circle of radius r, and L is the corresponding arc length, then $L = rs$.
- The area of a sector of a circle of radius r is $A_s = \frac{s}{2}r^2$.
- If (x,y) is the point on the unit circle that intersects the terminal side of an angle θ, then $\sin\theta = y$ and $\cos\theta = x$.



Chapter 2 review

[2–1] In each of the following sets of ordered pairs:

a. Determine if the set is a function.
b. If a function, state the domain and range.
c. If a one-to-one function, state the inverse function.

1. $\{(1,7), (4,5), (6,7), (7,10)\}$
2. $\{(-2,-3), (-1,1), (1,2), (2,5)\}$
3. $\{(1,3), (2,5), (2,9), (6,10)\}$

In each of the following problems a rule that describes a function is given. Form the ordered pairs that this function contains for the following domain elements:
a. -1 **b.** 0 **c.** $\sqrt{5}$ **d.** $\frac{1}{3}$

4. $f(x) = 4 - 2x$ **5.** $f(x) = 3x^2 - 2x + 3$

6. $f(x) = x^4 - 5$ **7.** $f(x) = \dfrac{3x}{x - 1}$

[2–2] In the following exercises draw the initial side and terminal side of the given angle. Also, state the measure of the smallest nonnegative angle that is coterminal with the angle.

8. 465° **9.** $-40.25°$ **10.** $-270°$
11. 1,800.6° **12.** 132°18′ **13.** 547°26′
14. 429.3° **15.** $-0°15'$

In the following problems you are given a point that lies on the terminal side of an angle in standard position. In each case draw a representation of the least positive angle that has the point on its terminal side and compute all six trigonometric functions for the angle.

16. $(5,-12)$ **17.** $(-5,8)$ **18.** $(0,-3)$
19. $(2,\sqrt{5})$ **20.** $(-\sqrt{2},3)$ **21.** $(1,-\sqrt{6})$
22. $(a,-2a)$, $a > 0$

[2–3] In the following problems you are given the sign of two of the trigonometric functions of an angle in standard position. State in which quadrant the angle terminates.

23. $\sin\theta < 0$, $\cos\theta < 0$ **24.** $\sec\theta < 0$, $\tan\theta > 0$
25. $\cos\theta > 0$, $\tan\theta < 0$ **26.** $\cot\theta < 0$, $\csc\theta < 0$
27. $\csc\theta > 0$, $\cot\theta < 0$ **28.** $\tan\theta < 0$, $\cos\theta < 0$
29. $\tan\theta > 0$, $\csc\theta < 0$ **30.** $\sec\theta > 0$, $\csc\theta < 0$

In the following problems you are given the degree measure of an angle in standard position. For each indicate the measure of the reference angle θ'.

31. 46.3° **32.** 323°16′ **33.** 421°48′
34. $-22.7°$ **35.** $-248.7°$ **36.** $-105.15°$
37. 242.57°

In the following problems you are asked to find a trigonometric function value for an angle. If the reference angle is 30°, 45°, or 60°, give the exact answer. Otherwise find the required value to four decimal places.

38. sin 213.4° **39.** cos 240°
40. tan(−114.6°) **41.** csc 870°
42. cot 212.3° **43.** tan 213.9°
44. sec 300° **45.** cos(−133°20′)
46. sin(−293°40′)

[2–4] In the following problems you are given a trigonometric function value of an angle in standard position, along with the sign of one of the other function values. Find the measure of the smallest nonnegative angle that meets these conditions, to the nearest 0.1°.

47. $\sin\theta = 0.3251$, $\cos\theta < 0$
48. $\cos\theta = -0.7771$, $\sin\theta > 0$
49. $\tan\theta = 0.6306$, $\sec\theta < 0$
50. $\sin\theta = -0.9088$, $\tan\theta < 0$
51. $\sec\theta = -2.0642$, $\sin\theta > 0$
52. $\cot\theta = 4.1046$, $\sin\theta < 0$
53. $\sin\theta = \dfrac{\sqrt{3}}{2}$, $\cos\theta < 0$
54. $\tan\theta = -2$, $\sin\theta < 0$
55. $\cos\theta = -\dfrac{5}{13}$, $\tan\theta < 0$

In the following problems you are given the value of one trigonometric function and the sign of another function of an angle in standard position. Draw a representation of the least positive angle that meets these conditions and use it. Compute the exact value of the remaining five trigonometric functions.

56. $\sin\theta = \frac{4}{5}$, $\cos\theta > 0$ **57.** $\cos\theta = -\frac{1}{2}$, $\tan\theta > 0$
58. $\cos\theta = -\frac{5}{16}$, $\tan\theta > 0$ **59.** $\cos\theta = \frac{1}{4}$, $\cot\theta < 0$
60. $\tan\theta = -2$, $\cos\theta < 0$ **61.** $\csc\theta = -4$, $\sec\theta > 0$
62. $\sec\theta = 6$, $\csc\theta < 0$ **63.** $\cot\theta = 2$, $\sin\theta < 0$
64. $\cot\theta = \frac{2}{7}$, $\sin\theta < 0$ **65.** $\tan\theta = \frac{7}{2}$, $\sec\theta < 0$
66. $\tan\theta = 0$, $\cos\theta > 0$ **67.** $\sin\theta = 0.25$, $\cos\theta < 0$

68. $\sin\theta = z$ and θ terminates in quadrant II. Find $\tan\theta$ in terms of z.

69. $\tan\theta = z$ and θ terminates in quadrant IV. Find $\cos\theta$ in terms of z.

70. In a certain electrical circuit the instantaneous voltage E (in volts) is found by the formula
$$E = 120\sin(\theta + 15°)$$
Compute E to the nearest 0.01 volt for the following values of θ:
a. 0° **b.** 45° **c.** 84.2° **d.** $-200°$
e. $-15°$ **f.** $-45°$

71. For a surveyor to locate a point by measuring an angle at one station and a distance from another one, the distance BP must be found by solving the following sequence of formulas:
$$\sin p = \frac{AB\sin b}{AP} \qquad (0° < p < 90°)$$
$$a = 180° - (b + p)$$
$$BP = \frac{AP\sin a}{\sin b}$$
Compute the distance BP to the nearest 0.1 meter if $AB = 211.5$ meters, $AP = 185.7$ meters, and $b = 29.6°$.

72. A machinist is setting up a numerically controlled drill. The drill must drill a hole in a piece of steel 9.0 millimeters from the origin at an angle of $125°30'$. To the nearest 0.1 millimeter, what are the coordinates of this point?

73. Suppose the hole of problem 72 must be 8.075 inches from the origin at an angle of $10.2°$. Find the coordinates of this point to the nearest 0.1 inch.

74. A technician is aligning a laser device used to cut patterns from cloth, and positions the device at an angle of $-42.3°$ and a distance of 6.90 feet from the origin. What should the x- and y-coordinates be at this point to the nearest 0.1 foot?

[2–5] Convert the following degree measures into radian measures. Leave your answers both in exact form and approximated to two decimal places.

75. $120°$ **76.** $-215°$ **77.** $430°$

Convert the following radian measures into degree measures. Leave your answers both in exact form and approximated to two decimal places.

78. $\frac{7\pi}{2}$ **79.** $\frac{11\pi}{3}$ **80.** $-\frac{5\pi}{7}$

81. 2.5 **82.** -4.2

83. Find the length of the arc determined by a central angle of 3.9 (radians) on a circle of diameter 6.3 inches, to the nearest 0.1 inch.

84. Find the measure, in radians, of a central angle on a circle of radius 8.8 mm determined by an arc length of 20 mm, to the nearest 0.1 radian.

85. Find the length of the arc determined by a central angle of $150°$ on a circle of diameter 15 mm.

86. The diameter of a wheel on an automobile is 30 inches. If the wheel moves through an angle of $385°$, how far will the car move?

Find the area of the sector determined by each of the following angles and radii. Give both the exact answer and a two-decimal-place approximation.

87. $30°$, 9 inches **88.** $240°$, 8 mm

89. $\frac{11}{12}$, 6 mm **90.** $\frac{2\pi}{5}$, 7 inches

[2–6] Find the following function values where the angle is given in radian measure. Round your answer to four decimal places.

91. $\sin 1.9$ **92.** $\sec 2.4$ **93.** $\tan 4.5$

Find the exact function values for the following angles.

94. $\sin \frac{5\pi}{3}$ **95.** $\tan \frac{3\pi}{4}$ **96.** $\cos \frac{7\pi}{6}$

97. $\cos\left(-\frac{\pi}{6}\right)$ **98.** $\tan\left(-\frac{4\pi}{3}\right)$ **99.** $\sin\left(-\frac{5\pi}{6}\right)$

100. $\sec\left(-\frac{\pi}{4}\right)$

Find the least nonnegative value of θ in radians. Round to two decimal places if necessary.

101. $\sin \theta = -\frac{1}{2}$, $\cos \theta > 0$
102. $\tan \theta = -1.82$, $\sin \theta > 0$

Find one solution to the following equations, in radians. Round to two decimal places if necessary.

103. $2 \sin \theta = 0.84$ **104.** $3 \tan 2\theta = 5$
105. $(2 \cos \theta - 1)(\cos \theta - 1) = 0$

106. The position d at the end of a spring, under certain initial conditions, as a function of time t in seconds, is

$$d = \tfrac{1}{3} \cos 8t - \tfrac{1}{4} \sin 8t$$

Compute d if $t = \frac{1}{12}$.

Chapter 2 test

1. Draw the initial side and terminal side of the given angle. Also, state the measure of the least nonnegative angle that is coterminal with the given angle.
a. $665°$ **b.** $-417°$

In the following problems you are given a point that lies on the terminal side of an angle in standard position. In each case draw a representation of the least positive angle that has the point on its terminal side and compute all six trigonometric function values for the angle (leave answers exact).

2. $(6, -12)$ **3.** $(-2, -6)$ **4.** $(\sqrt{2}a, a)$, $a > 0$

In the following problems you are given the value of one trigonometric function and the sign of another function of an angle in standard position. (a) Draw a representation of the least positive angle that meets these conditions. (b) Use a reference triangle to compute the exact value of the remaining five trigonometric functions. (c) Use one of the function values to find the least positive measure of the angle to the nearest 0.1°.

5. $\sin\theta = -\frac{1}{6}$, $\cos\theta > 0$ **6.** $\tan\theta = 8$, $\cos\theta < 0$

In the following problems you are given the degree measure of an angle in standard position. For each indicate the measure of the reference angle θ'.

7. 246.2° **8.** −55.6°

In the following problems you are asked to find a trigonometric function value for an angle. Find the required value to four decimal places.

9. sin 116.4° **10.** tan(−14.8°) **11.** csc 115°20′

In the following problems you are given a trigonometric function value of an angle in standard position, along with the sign of one of the other function values. Find the measure of the smallest nonnegative angle that meets these conditions to the nearest 0.1°.

12. $\sin\theta = -0.2961$, $\cos\theta < 0$
13. $\sec\theta = -2.0642$, $\sin\theta > 0$

14. In a certain electrical circuit the instantaneous current I is found by the formula $I = 5.4\cos(\theta - 25°)$. Compute I to the nearest 0.1 ampere for $\theta = 45°$.

15. The arm of an industrial robot is positioned at 22.6 centimeters from the origin at an angle of 261.42°. To the nearest 0.1 centimeter, what are the coordinates of this point?

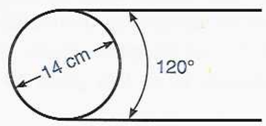
16. To align a precision laser that is part of an optical bench, a technician points the device at a test point with coordinates (−211.5, 620.0) (inches). Assuming the bench is coordinatized in the usual way, with the laser at the origin, (a) what angle should the laser's indicator show to the nearest 0.1°, and (b) how far from the origin is the test point?

17. Convert 415° into radian measure. Leave your answer both in exact form and approximated to two decimal places.

18. Convert $\frac{7\pi}{12}$ into degree measure.

19. Find the length of the arc subtended by a central angle of 5.0 (radians) on a circle of diameter 8.2 inches, to the nearest 0.1 inch.

20. The diameter of a wheel on a pulley is 14 cm (see the diagram). If the wheel moves through an angle of 120°, how far will the belt that the wheel drives move?



21. Find a four-decimal-place approximation to csc 2.5.

22. Find the exact value of $\cos\frac{5\pi}{6}$.

23. Find the area of the sector determined by a central angle of $\frac{\pi}{3}$ (radians) in a circle of diameter 32 mm, to the nearest 0.01 mm².

24. Let $f = \{(2,-3), (3,5), (5,6), (10,12)\}$.
- **a.** Is f a function?
- **b.** If so, is it one to one?
- **c.** Does f have an inverse? If so, state it.

25. If the function f is described by the rule $f(x) = x^2 - 3x + 5$, state the ordered pair that is an element of f when the domain element is (a) −4 and (b) $\sqrt{2}$.

Find the least nonnegative value of θ in radians. Round to two decimal places if necessary.

26. $\sin\theta = -\frac{1}{3}$, $\cos\theta > 0$
27. $\tan\theta = \sqrt{3}$, $\sin\theta < 0$

Find one solution to the following equations, in radians. Round to two decimal places if necessary.

28. $\frac{1}{2}\cos\theta = 0.20$ **29.** $2\tan 3\theta = 4.2$
30. $\sin\theta(2\sin\theta - 1) = 0$

31. The position d at the end of a spring, under certain initial conditions, as a function of time t in seconds, is $d = \frac{1}{3}\cos 4t - \frac{1}{4}\sin 4t$. Compute d if $t = \frac{\pi}{16}$.

GMAC Bank

Student Loans
for up to
$40,000 per year*

Defer payments until after graduation.**
Fast preliminary approval, usually in minutes.

Apply Now!
Go to gmacbankfunding.com

Apply online in as little as 15 minutes

- Loans up to $40,000 per academic year*
- Good for tuition and other educational expenses: books, fees, a laptop, room and board, travel home, etc.
- Get a check in as few as 5 business days
- Start payments now or up to six months after graduation**
- Reduce your interest rate by as much as 0.50% with automatic payments***

All loans are subject to application and credit approval.

* Undergraduate and graduate borrowers may borrow annually up to the lesser of the cost of attendance or $30,000 ($40,000 for certain schools where it has been determined that the annual cost of attendance exceeds $30,000). Borrowers in the Continuing Education loan program may borrow annually up to $30,000.

** Undergraduate students may choose to defer repayment until six months after graduation or ceasing to be enrolled at least half time in school. Interest only and immediate repayment options also available.

*** A 0.25% interest rate reduction is available for borrowers who elect to have monthly principal and interest payments transferred electronically from a savings or checking account. The interest rate reduction will begin when automatic principal and interest payments start, and will remain in effect as long as automatic payments continue without interruption. This reduced interest rate will return to contract rate if automatic payments are cancelled, rejected or returned for any reason. Upon request, borrowers are also entitled to an additional 0.25% interest rate reduction if (1) the first 36 payments of principal and interest are paid on time, and (2) at any time prior to the 36th on time payment, the borrower who receives the monthly bill elects to have monthly principal and interest payments transferred electronically from a savings or checking account, and continues to make such automatic payments through the 36th payment. This reduced interest rate will not be returned to contract rate if, after receiving the benefit, the borrower discontinues automatic electronic payment. The lender and servicer reserve the right to modify or discontinue borrower benefit programs (other than the co-signer release benefit) at any time without notice.